Resolving Singularities of Plane Algebraic Curves

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I am never forget the day I first meet the
Great Lobachevsky

Tom Lehrer
Lobachevsky

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Note: I have been allowed extra pages in this thesis.

CHAPTER 1

Introduction

> It's the job that's never started as takes longest to finish.
>
> *J.R.R. Tolkien*
> *The Lord of the Rings*

Algebraic geometry is a blending of linear algebra and differential geometry. Classical differential geometry studies the subsets of $\mathbb{R}^n$ arising as the intersections of zero sets of differentiable functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, whilst algebraic geometry is a generalisation of linear algebra in that it studies the simultaneous zeroes of a set I of polynomial functions of arbitrary degree over a field $\mathbb{K}$. These comprise the algebraic subsets $C = \zeta(I)$ of $\mathbb{K}^n$, we use $\zeta(I)$ to denote the set of simultaneous zeroes of the polynomials in the set I. We call such C an *algebraic variety*.

As polynomial functions are analytic, studying their zeroes makes algebraic geometry a specialisation of classical differential geometry. Polynomial functions can be defined for any field $\mathbb{K}$, so that the purely algebraic aspects apply to general fields. From an algebraic perspective, it is more convenient to work over an algebraically closed field, so even when primary interest is in zero sets of polynomials over $\mathbb{R}[x, y]$, called real plane curves, we work over $\mathbb{C}[x, y]$. We can recover the real plane curves by intersecting the complex plane curves with $\mathbb{R}^2$.

From the perspective of differentiable geometry, a topological space M is an n-dimensional differentiable manifold if every point p has an open neighbourhood homeomorphic (there exists a continuous function with continuous inverse) to $\mathbb{R}^n$ and can be assigned a well defined tangent (vector) space, $T_p(M)$, in such a way that it varies continuously with p. There are three ways in which this can fail to be the case at a point $p \in C$: 1) there is no open neighbourhood around p whose intersection with M is homeomorphic to $\mathbb{R}^n$, 2) T_pM is not defined, or 3) T_pM does not vary continuously with p.

Remark 1.0.1. It is possible for a point to fail all three conditions.

Example 1. Take $f(x, y) = y^2 - x^3 - x^2$, and let $C = \zeta(y^2 - x^3 - x^2)$. The point $(0, 0)$ does not have an open neighbourhood in the subspace topology which is homeomorphic to $\mathbb{R}^1$.

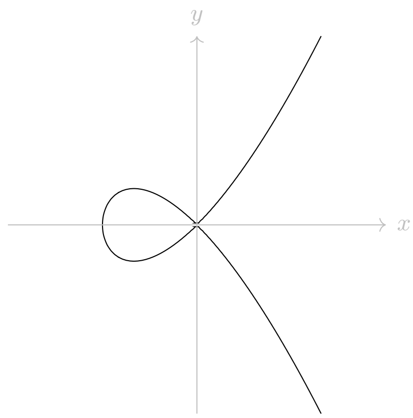


FIGURE 1. $y^2 - x^3 - x^2 = 0$

Example 2. Take $f(x, y) = y^2 - x^3$ and let $C = \zeta(f(x, y))$. The point $(0, 0)$ is a singularity as a velocity vector of a curve on C reverses direction as it passes through $(0, 0)$.

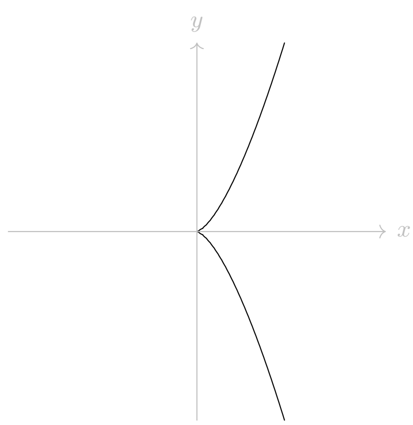


FIGURE 2. $y^2 - x^3 = 0$

Points at which the curve is a differentiable manifold are the *regular* points. The points where the curve fails to be a differentiable manifold are the *singular points*. We want a way of easily determining the singular points p of a curve C. From the examples, we consider singularities of a curve $C = \zeta(f(x, y))$ as points where C has multiple tangents. This can occur in two ways: firstly, the curve could have distinct tangents (Example 1), or it could have some tangents which coincide. There are multiple tangents at $(0, 0)$ if $f(x, y)$ has no linear term. Recall that the Taylor expansion of a polynomial f at $(0, 0)$ is just f, and so we generalise when a point p is a singularity by looking at the Taylor expansion of f around p.

Definition 1.0.2. Take $f \in \mathbb{C}[x,y]$, and let C be the curve defined by $f(x,y)$ in the complex plane $\mathbb{C}^2$.

1. The multiplicity of a point $p \in C$ is the order of the first non-vanishing term of the Taylor expansion of f around P, denoted $\nu_p(C)$.
2. If $\nu_p(C) = 1$, we call p a *regular* point of C.
3. If $\nu_p(C) > 1$, then p is a *singular* point of C.
4. If the number of distinct tangents at $p \in C$ is $\nu_p(C)$, and $\nu_p(C) > 1$, then p is an ordinary multiple point.
5. A curve C is non-singular if every point p of C is regular.

Remark 1.0.3. As C is the vanishing set of f, we start with $\nu_p(C) = 1$, not $\nu_p(C) = 0$.

Ideally, we want to study algebraic curves that are differentiable manifolds. As the examples above show, this is not always the case, and so we seek a way of studying these curves. We can do so by studying non-singular curves which are *equivalent*. That is, we want to resolve the singularities.

It is important to note that we do not seek curves which are isomorphic to the singular curve C, as these would still contain a singular point. Instead, we use the notion of *birational equivalence* (Definition 3.0.1) to find and study appropriate non-singular curves C', as this allows us to replace a singular point p in C with several points in C'.

When studying the zeroes of a polynomial f, we can think of the zeroes as the points of intersection between the curve $C = \zeta(f)$ and the line $y = 0$, the x-axis. We know that the number of roots of a polynomial f, counting multiplicity, is an invariant for polynomials of degree n, and so we seek an analogous notion of multiplicity for the intersection of curves. The first step in obtaining such an invariant for two curves, is ensuring that they always intersect. Take a pair of parallel lines in $\mathbb{C}^2$: their intersection is empty, and so it seems that our search has failed at the simplest of examples.

However, this is easily overcome: the intersection of two non-parallel lines ℓ_1 and ℓ_2 in $\mathbb{C}^2$ consists of a single point p. Yet, as we perturb ℓ_1 so that it becomes parallel to ℓ_2, we see that the intersection point p *drifts away to infinity*.

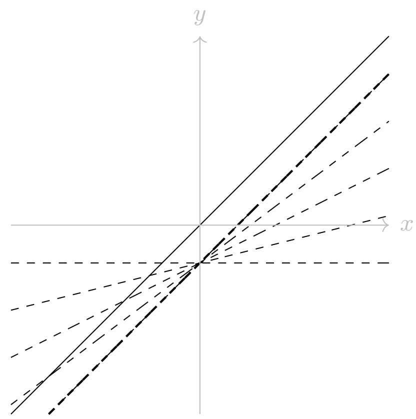


FIGURE 3. $y^2 - x^3 = 0$

Thus, we introduce the notion of a *point at infinity*. This is equivalent to considering the lines in the projectivisation $\mathbb{CP}^2$ of $\mathbb{C}^2$. Complex projective n-space, $\mathbb{CP}^n$, comprises the set of all lines through the origin (one-dimensional vector subspaces) of $\mathbb{C}^{n+1}$, with the quotient topology induced from $\mathbb{C}^{n+1} \setminus \{(0, \ldots, 0)\}$ by the equivalence relation $(x_0, \ldots x_n) \sim (u_0, \ldots, u_n)$ if and only if there is a λ with $u_i = \lambda x_i$ for $i = 0, \ldots, n$. We write $[x_0 : \cdots : x_n]$ for the equivalence class of $(x_0, \ldots x_n)$, and call these *homogeneous coordinates*.

For each $j \in \{1, \ldots, n\}$, the map

$$\varphi_j \colon \mathbb{C}^n \longrightarrow \mathbb{CP}^n, \quad (u_1, \ldots, u_n) \longmapsto [x_0 : \cdots : x_n]$$

with

$$x_i = \begin{cases} u_{i+i} & \text{for } i < j \\ 1 & \text{for } i = j \\ u_i & \text{for } i > j \end{cases}$$

imbeds $\mathbb{C}^n$ homeomorphically as a subset of $\mathbb{CP}^n$.

So, even when the prime interest is in algebraic subsets of $\mathbb{C}^n$, it is convenient to embed these in, and to work with, complex projective n-space $\mathbb{CP}^n$. In our case, while we focus on complex plane curves, algebraic varieties in $\mathbb{C}^2$ which are the zero sets of a polynomial $f \in \mathbb{C}[x, y]$, we study them in $\mathbb{CP}^2$. For ease, examples and diagrams will be over $\mathbb{R}^2$.

There are several ways to resolve singularities. This thesis discusses resolving singularities by repeatedly "blowing up" the singularity (Section 3.1). Before discussing blow ups, we discuss intersections of curves, as we examine the intersection of two curves at a singularity p to show that we obtain an non-singular curve in a finite number of blow ups. We discuss the use of elimination theory to find common zeroes of two polynomials $f(x, y)$ and $g(x, y)$, and then introduce the *intersection multiplicity* of two curves at a point. In the final section, we discuss a method of expressing solutions y of polynomial equation $f(x,y) = 0$ as a (fractional) power series in x, by considering the *Newton polygon* of $f(x,y)$. The fractional power series, *Puiseux expansion*, and the Newton polygon are used to examine the blow ups of singular points.

Chapter 3 starts by defining the blowing up a point of a curve, and to ensure we obtain another plane curve, we introduce σ-processes (Definition 3.1.6), before examining the behaviour of the intersection of C and its auxiliary equations at the singularity under σ-processes. We also use intersection multiplicity to define a standard resolution of a singularity (Definition 3.1.10). This will allow us to show that we can resolve a singular point of any irreducible curve C through a finite sequence of σ-processes (Theorem 3.2.19). If the singularity is sufficiently complex, performing a sequence of blow ups becomes inefficient and arduous, so we discuss an efficient algorithm (Section 3.3). This algorithm, however, has technical limitations which can be overcome using transformations of $\mathbb{CP}^2$, called quadratic transformations. In Section 3.4 we extend σ-processes to resolve several singular points of an *irreducible* curve C, Corollary 3.4.29. In Section 3.5, we remove the irreducibility assumption, extending our results to *reducible* curves with Theorem 3.5.31. In Chapter 4 we examine two methods of representing the standard resolution of a singularity, namely the multiplicity sequence and the resolution graph. We first introduce the multiplicity sequence (Section 4.1), and then resolution graph (Section 4.2). In the last section, we show that the information continued in the Puiseux characteristic exponents (Definition 2.4.20), multiplicity sequence, and resolution graph is equivalent.

CHAPTER 2

Intersections of Curves

> Lines, the straightest of curves! Order yours today!
>
> *Ross Ogilvie*

In order to understand the effect of σ-processes on a singularity, we examine how they affect curves which intersect at the singularity. This means we need to understand how two curves intersect at a point.

Finding points of intersections of two curves $C = \zeta(f(x, y))$ and $D = \zeta(g(x, y))$ is equivalent to finding common zeroes of $f(x, y)$ and $g(x, y)$. We can find common zeroes of $f(x, y)$ and $g(x, y)$ by eliminating powers of y, and obtaining an equation $R(x)$ in x. By finding the roots of $R(x)$, we obtain candidate x-values of common zeroes of $f(x, y)$ and $g(x, y)$. The process of eliminating powers of y can be represented by a matrix called the Sylvester matrix (Definition 2.1.1). We show that if (α, β) is a common zero of f and g, then $x = \alpha$ is a root of the determinant, $R_{f,g}(x)$, of the Sylvester matrix. This provides us with an initial method for interpreting the *multiplicity* of (α, β) as a common zero of f and g through the multiplicity of $x = \alpha$ as a root of $R_{f,g}(x)$. There are however, issues with this interpretation, and so we introduce a geometric description of the intersection of two curves, called the *intersection number* (Definition 2.2.7), which avoids these concerns. We the show that the intersection number and intersection multiplicity are equal, and so the intersection multiplicity is also well defined.

Having seen a geometric and an algebraic way of interpreting the multiplicity of points of intersection, we show that there is also a geometric way of looking at the multiplicity of a point on a curve, and show that it agrees with our original definition.

Finally, we examine how to express a solution y of $f(x, y) = 0$ in terms of x, as this will allow us to compute the common zeroes of two polynomials, or the intersection points of two curves. We introduce *Newton polygons* as a tool for decomposition the polynomial $f(x, y)$ into components for which we know how to find an expression of y in terms of x. Newton polygons will also play an important role in Chapter 3, when we show that we can *resolve* a singularity in a finite number of steps.

2.1. Finding Points of Intersection

One way of finding common zeroes of two polynomials $f(x, y)$ and $g(x, y)$, is to find an equation for the x-values of the common zeroes, by eliminating powers of y. We consider polynomials in $\mathbb{C}[x, y]$ as elements of $\mathbb{C}[x][y]$, that is, polynomials in y with coefficients in $\mathbb{C}[x]$.

Example 3. We seek the common zeroes of polynomials f and g of degrees 3 and 1 in y respectively.

Take

$$f(x, y) = a_0(x) + a_1(x)y + a_2(x)y^2 + a_3(x)y^3 = 0 \tag{21}$$

$$g(x, y) = b_0(x) + b_1(x)y = 0 \tag{22}$$

We subtract $y^2 a_3(x)g(x, y)$ from $b_1(x)f(x, y)$ to eliminate y^3:

$$b_1(x)a_0(x) + b_1(x)a_1(x)y + (b_1(x)a_2(x) - a_3(x)b_0(x))y^2 = 0 \tag{23}$$

We now eliminate y^2 from (23), by subtracting $(b_1(x)a_2(x) - a_3(x)b_0(x))y$ times (22) from $b_1(x)$ times (23):

$$b_1(x)b_1(x)a_0(x) + (b_1(x)b_1(x)a_1(x) - b_0(x)b_1(x)a_2(x) - b_0(x)a_3(x)b_0(x))y = 0 \tag{24}$$

Next, we eliminate y from (24) by subtracting $(b_1(x)b_1(x)a_1(x) - b_0(x)b_1(x)a_2(x) - b_0(x)a_3(x)b_0(x))$ times (22) from $b_1(x)$ times (24)

(25)
$$b_1(x)b_1(x)b_1(x)a_0(x) - b_0(x)b_1(x)b_1(x)a_1(x) + b_0(x)b_0(x)b_1(x)a_2(x) + b_0(x)b_0(x)a_3(x)b_0(x) = 0$$

Thus, a necessary condition for (x, y) to be a common zero of f and g is that x be a zero of (25). This is equivalent to x being such that the following system of simultaneous equations in the unknowns $1, y, y^2, y^3$ has non-trivial solutions:

$$\begin{array}{llllll}
f(x, y) & = a_0 & + a_1 y & + a_2 y^2 & + a_3 y^3 & = 0 \\
g(x, y) & = b_0 & + b_1 y & + 0 & + 0 & = 0 \\
y g(x, y) & = 0 & + b_0 y & + b_1 y^2 & + 0 & = 0 \\
y^2 g(x, y) & = 0 & + 0 & + b_0 y^2 & + b_1 y^3 & = 0
\end{array}$$

This system has non-trivial solutions when the determinant of the coordinate matrix

$$\begin{bmatrix} a_0(x) & a_1(x) & a_2(x) & a_3(x) \\ b_0(x) & b_1(x) & 0 & 0 \\ 0 & b_0(x) & b_1(x) & 0 \\ 0 & 0 & b_0(x) & b_1(x) \end{bmatrix}$$

is 0.

Example 4. Take

$$f(x,y) = a_0(x) + a_1(x) + a_2(x)y^2 + a_3(x)y^3 + a_4(x)y^4 + a_5(x)y^5 = 0 \tag{26}$$

$$g(x,y) = b_0(x) + b_1(x)y + b_2(x)y^2 + b_3(x)y^3 = 0 \tag{27}$$

of degrees 5 and 3 respectively.

We seek a necessary condition on x for (x, y) to be a common zero of f and g. That is, the values of x such that there are non-trivial solutions to the system of equations in $1, y, y^2, y^3, y^4, y^5$:

$$\begin{array}{llllllll} f(x,y) & = a_0 & + a_1 y & + a_2 y^2 & + a_3 y^3 & + a_4 y^4 & + a_5 y^5 & = 0 \\ g(x,y) & = b_0 & + b_1 y & + b_2 y^2 & + b_3 y^3 & + 0 & + 0 & = 0 \\ yg(x,y) & = 0 & + b_0 y & + b_1 y^2 & + b_2 y^3 & + b_3 y^4 & + 0 & = 0 \\ y^2 g(x,y) & = 0 & + 0 & + b_0 y^2 & + b_1 y^3 & + b_2 y^4 & + b_3 y^5 & = 0 \end{array}$$

This is system of four equations in six unknowns. To eliminate powers of y, we introduced additional polynomials whose common zeroes with f and g are the common zeroes of f and g. The additional polynomials are products of $g(x, y)$ and powers of y. We can also multiply $f(x, y)$ by powers of y, introducing additional powers of y to be eliminated. For example, $yf(x, y)$ introduces a y^6 term, which we can eliminate using $y^3 g(x, y)$.

While we have introduced a new power of y, we also obtain two new equations, yielding a system of six equations in seven unknowns. Repeating this with $y^2 f(x, y)$ and $y^4 g(x, y)$ results in a system of eight equations in eight unknowns:

$$\begin{array}{llllllllll} a_0 & + a_1 y & + a_2 y^2 & + a_3 y^3 & + a_4 y^4 & + a_5 y^5 & + 0 & + 0 & = f(x,y) & = 0 \\ 0 & + a_0 y & + a_1 y^2 & + a_2 y^3 & + a_3 y^4 & + a_4 y^5 & + a_5 y^6 & + 0 & = yf(x,y) & = 0 \\ 0 & + 0 & + a_0 y^2 & + a_1 y^3 & + a_2 y^4 & + a_3 y^5 & + a_4 y^6 & + a_5 y^7 & = y^2 f(x,y) & = 0 \\ b_0 & + b_1 y & + b_2 y^2 & + b_3 y^3 & + 0 & + 0 & + 0 & + 0 & = g(x,y) & = 0 \\ 0 & + b_0 y & + b_1 y^2 & + b_2 y^3 & + b_3 y^4 & + 0 & + 0 & + 0 & = yg(x,y) & = 0 \\ 0 & + 0 & + b_0 y^2 & + b_1 y^3 & + b_2 y^4 & + b_3 y^5 & + 0 & + 0 & = y^2 g(x,y) & = 0 \\ 0 & + 0 & + 0 & + b_0 y^3 & + b_1 y^4 & + b_2 y^5 & + b_3 y^6 & + 0 & = y^3 g(x,y) & = 0 \\ 0 & + 0 & + 0 & + 0 & + b_0 y^4 & + b_1 y^5 & + b_2 y^6 & + b_3 y^7 & = y^4 g(x,y) & = 0 \end{array}$$

This system has non-trivial solutions when the determinant of

$$\begin{bmatrix} a_0(x) & a_1(x) & a_2(x) & a_3(x) & a_4(x) & a_5(x) & 0 & 0 \\ 0 & a_0(x) & a_1(x) & a_2(x) & a_3(x) & a_4(x) & a_5(x) & 0 \\ 0 & 0 & a_0(x) & a_1(x) & {}_2(x) & a_3(x) & a_4(x) & a_5(x) \\ b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 & 0 & 0 & 0 \\ 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 & 0 & 0 \\ 0 & 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 & 0 \\ 0 & 0 & 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 \\ 0 & 0 & 0 & 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) \end{bmatrix}$$

is 0. This determinant is a polynomial in x, and so we have a necessary condition on the values of x: the x-values of common zeroes must be roots of the determinant of the coefficient matrix.

Consequently, we make the following definition.

Definition 2.1.1. Given two polynomials $f, g \in \mathbb{C}[x, y]$

$$f(x, y) = \sum_{i=0}^{n} a_i(x) y^i$$

$$g(x, y) = \sum_{j=0}^{m} b_j(x) y^j$$

We call the $(m + n) \times (m + n)$ matrix

$$\begin{bmatrix} a_0 & a_1 & \dots & \dots & \dots & a_n & 0 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & \dots & \dots & \dots & a_n & 0 & 0 & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & a_0 & a_1 & \dots & \dots & a_{n-1} & a_n & 0 \\ 0 & 0 & 0 & 0 & a_0 & a_1 & \dots & \dots & a_{m-1} & a_m \\ b_0 & b_1 & \dots & \dots & \dots & b_{m-1} & b_m & 0 & 0 & 0 \\ 0 & b_0 & b_1 & \dots & \dots & \dots & b_{m-1} & b_m & 0 & 0 \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & & \\ 0 & 0 & b_0 & b_1 & \dots & \dots & \dots & b_{m-1} & b_m & 0 \\ 0 & 0 & 0 & b_0 & b_1 & \dots & \dots & \dots & b_{m-1} & b_m \end{bmatrix}$$

the Sylvester Matrix of f and g, and its determinant the resultant of f and g, denoted $R_{f,g}(x)$.

Example 5. Take the polynomials $f(x, y) = x^2 - y$ and $g(x, y) = -x^2 - y$. Their resultant is

$$R_{f,g} = \det \begin{bmatrix} x^2 & -1 \\ -x^2 & -1 \end{bmatrix} = -2x^2 = -2(x-0)(x-0)$$

So the common zero is $(0, 0)$.

Now, take $h(x, y) = -x + y$, we consider the common zeroes of h and f.

Their resultant is

$$R_{f,h} = \det \begin{bmatrix} x^2 & -1 \\ -x & +1 \end{bmatrix} = x^2 + x = x(x-1)$$

And the common zeroes are $(0, 0)$ and $(1, 1)$.

Remark 2.1.2. If we were to construct a notion of a common zero (α, β) of two polynomials occurring twice, then supposing we have an equation which gives us the x-values, it should give us $x = \alpha$ twice. Indeed we have such an equation, namely the resultant. So we could propose to interpret the multiplicity of the common zero (α, β) of f and g as the multiplicity of α as a zero of $R_{f,g}(x)$ [1].

There are two issues with this proposal. Firstly, we could have eliminated powers of x instead of y from the polynomials, obtaining a resultant in y, $R_{f,g}(y)$. Given a common zero (x, y), the multiplicity of y as a root of $R_{f,g}(y)$ should correspond to the multiplicity of x as a root of $R_{f,g}(x)$. Secondly, given two common zeroes with the same x-coordinate α, but different y-coordinates β_1 and β_2, the resultant will not distinguish the two. The multiplicity of $x = \alpha$ as a root of $R_{f,g}(x)$ will be the sum of the multiplicities of (α, β_1) and (α, β_2).

Thus, to be able to have a sensible definition of the *multiplicity* of common zeroes, we need to find a way which is independent of choices of coordinates.

Returning to considerations of curves, we have found a way of describing the intersection of two curves $C = \zeta(f(x, y))$ and $D = \zeta(g(x, y))$ at a point (α, β) through the multiplicity of the root $x = \alpha$ of $R_{f,g}(x)$, which appears dependent on choice of coordinates.

2.2. Intersection Multiplicity

We begin our search for a the notion of the intersection multiplicity of two curves C and D at a point p, independent of choice of coordinates. We find a way around these issues by looking at the multiplicity of roots of a polynomial from a new perspective [6].[1]

[1] I would like to thank Professor John Rice for the helpful conversations about intersection multiplicity, and his unpublished notes.

Given a set $A = \{a_1, \ldots, a_k\}$, we consider the evaluation map from $\mathbb{C}[x]$ to $\mathbb{C}^k$:

$$\begin{aligned} \text{Eval}_A : \mathbb{C}[x] &\longrightarrow \mathbb{C}^k \\ q(x) &\longmapsto \begin{bmatrix} q(a_1) \\ \cdots \\ q(a_k) \end{bmatrix}. \end{aligned}$$

Remark 2.2.3. We can consider $\mathbb{C}^k$ in this role as the space of functions on the set A, and the dimension of $\mathbb{C}^k$ is equal to the number of points in A.

We will see that Eval_A is a surjection. The kernel of Eval_A is the set of polynomials $f \in \mathbb{C}[x]$ such that $f(a_i) = 0$ for each $a_i \in A$.

If $f(a_i) = 0$, then $(x - a_i)$ divides $f(x)$, so given $f(x) \in \ker(\text{Eval}_A)$, $(x - a_i)$ divides $f(x)$ for $1 \leq i \leq k$. Thus

$$p(x) = \prod_{i=1}^{k}(x - a_i)$$

also divides $f(x)$, and $p(x)$ generates the kernel of Eval_A.

Hence, we have an isomorphism between $\mathbb{C}^k$ and

$$\mathbb{C}[x] \big/ p(x)\mathbb{C}[x],$$

and we can think of this quotient as the space of polynomials with domain A. By Remark 2.2.3, the dimension of the quotient ring is equal to the cardinality of A.

We now impose this train of thought to consider any quotient of $\mathbb{C}[x]$ as a space of functions on a domain whose cardinality is the dimension of the quotient.

Example 6. Take $p(x) = x^n$. Then the space of functions we are considering consists of polynomials $f(x) \in \mathbb{C}[x]$ truncated from x^n onwards. The root of x^n is 0, but the dimension of the quotient space is n.

We can reconcile this by thinking of the quotient by x^n as the restriction of polynomials to a domain which contains n copies of 0, and we think of these as being ‘infinitely close’. We call this domain the $n - 1$ ‘infinitesimal neighbourhood’ of 0. Thus, the dimension of the quotient counts the number of points in this infinitesimal neighbourhood, namely 0 repeated n times.

In the same manner, we consider any quotient of $\mathbb{C}[x]$ by the polynomial $(x - a_i)^{n_i}$ as the space of polynomials restricted a domain with n_i copies of a_i, which we think of as an ‘infinitesimal neighbourhood’ of a_i. So given a general polynomial

$$p(x) = \prod_{i=1}^{k} (x - a_i)^{n_i},$$

we want to think of

$$\mathbb{C}[x] \big/ p(x)\mathbb{C}[x]$$

as the restriction of polynomials to the some 'infinitesimal neighbourhoods' of its roots.

Note, that for each i we have the canonical homomorphism

$$\eta_i : \mathbb{C}[x] \longrightarrow \mathbb{C}[x] \big/ (x - a_i)^{n_i}\mathbb{C}[x],$$

which restricts polynomials to the $n_i - 1$ infinitesimal neighbourhood of a_i.

We also have the homomorphism

$$\tilde{\eta}_i : \mathbb{C}[x] \big/ p(x)\mathbb{C}[x] \longrightarrow \mathbb{C}[x] \big/ (x - a_i)^{n_i}\mathbb{C}[x],$$

which we can also think of restriction polynomials to the $n_i - 1$ infinitesimal neighbourhood of a_i. This $\tilde{\eta}_i$ is just a further restricting elements of

$$\mathbb{C}[x] \big/ p(x)\mathbb{C}[x].$$

We can compose the canonical projection map to

$$\mathbb{C}[x] \big/ p(x)\mathbb{C}[x]$$

with $\tilde{\eta}_i$:

$$\xi : \mathbb{C}[x] \longrightarrow \mathbb{C}[x] \big/ p(x)\mathbb{C}[x] \longrightarrow \mathbb{C}[x] \big/ (x - a_i)^{n_i}\mathbb{C}[x],$$

Remark 2.2.4. The homomorphisms η_i and ξ are equal.

This allows us to partition

$$\mathbb{C}[x] \big/ p(x)\mathbb{C}[x]$$

into polynomials on the $n_i - 1$ infinitesimal neighbourhoods of the a_i. In fact, we can construct a homomorphism

$$\mathcal{A} : \mathbb{C}[x] \longrightarrow \bigoplus_{i=1}^{i=k} \mathbb{C}[x] \big/ (x - a_i)^{n_i}\mathbb{C}[x]$$

from the η_i.

The kernel of $\mathcal{A}$ consists of polynomials $f(x)$ which are divisible by each $(x-a_i)^{n_i}$. As $(x-a_i)^{n_i}$ and $(x-a_j)^{n_j}$ are relatively prime for $i \neq j$, and their product divides $f(x)$, $p(x)$ divides $f(x)$, and $\ker(\text{Eval}_A = p(x)\mathbb{C}[x]$.

Further, $(x-a_i)^{n_i}$ and $\prod_{j\neq i1}(x-a_j)^{n_j}$ are relatively prime, so there are polynomials $r_i(x)$ and $s_i(x)$ such that

$$r_i(x)(x-a_i)^{n_i} + s_i(x)\prod_{j\neq i1}(x-a_j)^{n_j} = 1.$$

Thus, given any element of

$$\bigoplus_{i=1}^{i=k} \mathbb{C}[x] \Big/ (x-a_i)^{n_i}\mathbb{C}[x]$$

we can construct a polynomial whose image is this element.

Remark 2.2.5. In the case $n_i = 1$ for each i, we have shown that Eval_A is a surjection.

Remark 2.2.6. This is the Chinese Remainder Theorem.

Thus, we have

$$\mathbb{C}[x] \Big/ p(x)\mathbb{C}[x] \cong \bigoplus_{i=1}^{i=k} \mathbb{C}[x] \Big/ (x-a_i)^{n_i}\mathbb{C}[x],$$

and we think of

$$\mathbb{C}[x] \Big/ p(x)\mathbb{C}[x]$$

as the restriction of polynomials to the $n_i - 1$ infinitesimal neighbourhoods of the a_i.

We now seek an analogous interpretation of quotients of $\mathbb{C}[x,y]$, and in doing so develop a notion of *multiplicity* for the common zeroes of two polynomials $f(x,y)$ and $g(x,y)$ in $\mathbb{C}[x,y]$. That is, we want a way of thinking of

$$\mathbb{C}[x,y] \Big/ J\mathbb{C}[x,y]$$

as the restriction of polynomials to some infinitesimal neighbourhoods of points in $\mathbb{C}^2$. First, we must find ideals J of $\mathbb{C}[x,y]$, for which we can interpret

$$\mathbb{C}[x,y] \Big/ J$$

as the restriction of polynomials to a point.

Given a point $(\alpha,\beta) \in \mathbb{C}[x,y]$, the ideal of polynomials that are zero at (α,β) is the ideal $\langle x-\alpha, y-\beta\rangle$, which is a finitely generated maximal ideal of $\mathbb{C}[x,y]$. So, we can think of

$$\mathbb{C}[x,y] \Big/ J\mathbb{C}[x,y]$$

as the restriction of polynomials to the point (α, β).

Now consider ideals I generated by powers of $(x - \alpha)$ and $(y - \beta)$. These are contained in $J = \langle x - \alpha, y - \beta \rangle$, and are also finitely generated. By the Hilbert Nullstellensatz (Section 1.7 [2]), each such ideal $I = \langle (x - \alpha)^d, (y - \beta)^d \rangle$ contains a power of J: there is some k such that $J^k \leq I$. Hence, we have a map of quotients

$$\mathbb{C}[x,y] \Big/ J^k\mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y] \Big/ I\mathbb{C}[x,y].$$

As J^d is finitely generated, the dimension of

$$\mathbb{C}[x,y] \Big/ J^d\mathbb{C}[x,y]$$

is finite, and thus so is the dimension of

$$\mathbb{C}[x,y] \Big/ I\mathbb{C}[x,y].$$

We consider the quotient by I to be the restriction of polynomials to a set A_I, such that number of points in the A_I is the dimension of this quotient.

For

$$\mathbb{C}[x,y] \Big/ J^k\mathbb{C}[x,y],$$

we think of A_J as consisting of k copies of (α, β).

Recall we took $I = \langle (x - \alpha)^d, (y - \beta)^d \rangle$, and so by our train of thought, A_I consists of d points with x-coordinate infinitely near α, and d points with y-coordinate infinitely near β. That is, it is the product of the $d - 1$ infinitesimal neighbourhood of $x = \alpha$ and the $d - 1$ infinitesimal neighbourhood of $y = \beta$, which we think of the $d - 1$ infinitesimal neighbourhood of (α, β): it consists of d copies of (α, β).

Hence, we take the dimension of

$$\mathbb{C}[x,y] \Big/ I\mathbb{C}[x,y]$$

to be the *multiplicity* of the common zero (α, β) of $(x - \alpha)^d$ and $(y - \beta)^d$.

Generalising, we take two polynomials $f(x, y)$ and $g(x, y)$ with no common component, so the set of common zeroes p_i is finite. Now, consider the maximal ideal J_i at each p_i. The intersection of the ideals J_i is the set of polynomials which vanish at each p_i, and so is contained in the ideal $\langle f, g \rangle$.

Now, each J_i is a maximal ideal of $\mathbb{C}[x, y]$ and so J_i and J_l are relatively prime for $i \neq l$. Thus, the intersection of the J_i is equal to their product.

By the Hilbert Nullstellensatz again, there is a d such that

$$\left(\prod J_i\right)^d \leq \langle f, g\rangle.$$

We will use this to develop a *partition* of $\langle f, g\rangle$ amongst the common zeroes p_i. After which we think of

$$\mathbb{C}[x,y] \Big/ \langle f, g\rangle\mathbb{C}[x,y]$$

as the restriction of polynomials to infinitesimal neighbourhoods of each p_i.

Consider the ideal $I_i := \langle f, g\rangle + J_i^d$, both $\langle f, g\rangle$ and J^d are contained in the maximal ideal J, and so I is also contained in J. But as I_i contains J_i^d, a power of a maximal ideal, I_i is not contained in any other maximal ideal of $\mathbb{C}[x, y]$. Thus, we can use the ideals I_i to partition $\langle f, g\rangle$ according to the common zeroes p_i of f and g, as desired.

We have the map of quotients

$$\mathbb{C}[x,y] \Big/ J_i^d\mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y] \Big/ I_i\mathbb{C}[x,y],$$

and so by the same argument, the dimension of

$$\mathbb{C}[x,y] \Big/ I_i\mathbb{C}[x,y]$$

is finite. Hence, we take this dimension as the multiplicity of p_i as a common zero of $f(x, y)$ and $g(x, y)$.

Definition 2.2.7. Given two polynomials $f(x, y), g(x, y) \in \mathbb{C}[x, y]$, the multiplicity of a common root $(\alpha_i\beta_i)$ is the dimension of the quotient

$$\mathbb{C}[x,y] \Big/ I_i$$

Recall that the common zeroes of $f(x, y)$ and $g(x, y)$ are the intersection points of the curves $C = \zeta(f(x, y))$ and $D = \zeta(g(x, y))$.

So, given two curves $C = \zeta(f)$ and $D = \zeta(g)$ in $\mathbb{C}^2$ with $f, g \in \mathbb{C}[x, y]$, the intersection multiplicity $\nu_{p_i}(C, D)$ of C and D at (α_i, β_i) is the multiplicity of (α_i, β_i) as a common zero of f and g.

Remark 2.2.8. If f and g arise from homogenous polynomials F and G, of degree n and m respectively, by setting $z = 1$, and no intersection points occur when $z = 0$, then

$$\dim_{\mathbb{C}} \left(\mathbb{C}[x,y] \Big/ \langle f, g\rangle\mathbb{C}[x,y]\right) = nm,$$

and the left hand side is equal to the sum of the multiplicities of the points of intersection. This shows that when considered in $\mathbb{CP}^2$, the sum of the intersection multiplicities is equal to nm, the product of the degrees of f and g. This is a classical result, called Bezout's Theorem.

Remark 2.2.9. With the concept of localisation at a prime,

$$\mathbb{C}[x,y] \big/ J_i$$

is the localisation of the quotient by $\langle f, g\rangle$ at the common zero p_i,

$$\left(\mathbb{C}[x,y] \big/ \langle f,g\rangle\mathbb{C}[x,y]\right)_{(\alpha_i,\beta_i)} = \mathbb{C}[x,y]_{(\alpha_i,\beta_i)} \big/ \langle f,g\rangle\mathbb{C}[x,y]_{(\alpha_i,\beta_i)}.$$

Thus, the dimension of

$$\mathbb{C}[x,y]_{(\alpha_i,\beta_i)} \big/ \langle f,g\rangle\mathbb{C}[x,y]_{(\alpha_i,\beta_i)}$$

is the multiplicity of p_i as a common zero of f and g. For those unfamiliar with localisation, we introduce localisation of rings, and show that this holds.

We first define the localisation of the ring R and a multiplicative subset S, a subset of R which is closed under multiplication and contains the identity element of R [5].

Definition 2.2.10. Let R a commutative ring, and S a multiplicative subset of R. The ring

$$S^{-1}R := \{s^{-1}r \mid s \in S, r \in R\}$$

is the localisation of R at S.

For our specific case, we are localising at a point p, which is the same as considering the rational functions defined at p, which form a ring [2]:

Definition 2.2.11. Given a curve $C \subset \mathbb{C}$, and $p \in C$, the ring

$$\mathcal{O}_p(C) = \left\{ \frac{f}{g} \,\middle|\, f, g \in \mathbb{C}[x],\ g(p) \neq 0 \right\}$$

is called the local ring of C at p. And the ideal

$$\mathfrak{m}_p(C) = \{f \in \mathcal{O}_p(C) \mid f(p) = 0\}$$

is the unique maximal ideal of $\mathcal{O}_p$, called the maximal ideal of C at p.

We can now state the Lemma which shows that

$$\left(\mathbb{C}[x,y] \big/ \langle f,g\rangle\mathbb{C}[x,y]\right)_{(\alpha_i,\beta_i)} = \mathbb{C}[x,y]_{(\alpha_i,\beta_i)} \big/ \langle f,g\rangle\mathbb{C}[x,y]_{(\alpha_i,\beta_i)},$$

and thus the intersection multiplicity of C and D at the point p is equal to the dimension of the localisation at p of quotient of $\mathbb{C}[x,y]$ by the ideal $\langle f, g\rangle$.

Lemma 2.2.12. Given a polynomial $p(x) = \prod_{i=1}^{j}(x - a_i)^{n_i}$ the localisation of the $\mathbb{C}[x]$ module

$$\mathbb{C}[x] \Big/ p(x)\mathbb{C}[x]$$

at $x = a_i$ is

$$\mathbb{C}[x] \Big/ (x - a_i)^{n_i}\mathbb{C}[x]$$

and is zero if we are localising at a point which is not a root of the polynomial.

Proof. Consider the short exact sequence

$$0 \longrightarrow p(x)\mathbb{C}[x] \longrightarrow \mathbb{C}[x] \longrightarrow \mathbb{C}[x] \Big/ p(x)\mathbb{C}[x] \longrightarrow 0$$

Localisation at a preserves short exact sequences, and thus

$$0 \longrightarrow p(x)\mathbb{C}[x]_a \longrightarrow \mathbb{C}[x]_a \longrightarrow \left(\mathbb{C}[x] \Big/ p(x)\mathbb{C}[x]\right)_a \longrightarrow 0$$

is also a short exact sequence.

Consider $p(x)\mathbb{C}[x]_a$: it consists of elements $\frac{fp}{g}$ with $g(a) \neq 0$. If a is a root of $p(x)$ with multiplicity n, then $p(x)\mathbb{C}[x]_a$ is the space of polynomials which are zero on a, and thus

$$\left(\mathbb{C}[x] \Big/ p(x)\mathbb{C}[x]\right)_a = \mathbb{C}[x] \Big/ (x - a)^n\mathbb{C}[x].$$

If a is not a root of $p(x)$, we can take $g(x) = p(x)$, and so $p(x)\mathbb{C}[x]_a = \mathbb{C}[x]$, and hence

$$\left(\mathbb{C}[x] \Big/ p(x)\mathbb{C}[x]\right)_a$$

is zero. □

We now show that Definition 2.2.7 is equal to the notion of intersection multiplicity we considered in Remark 2.1.2. That is, we are seeking a geometric interpretation of the roots of the resultant $R_{f,g}(x)$.

To have such an interpretation, we construct a $\mathbb{C}[x]$ module homomorphism related to the resultant. Let $\mathbb{C}_d[x, y]$ be the vector space of polynomials in x and y of degree less than d in y with the basis $\{1, y, y^2, \ldots, y^d\}$. The Sylvester matrix of f and g from Definition 2.1.1 represents a $\mathbb{C}[x]$ module homomorphism

$$\begin{aligned} \chi_{f,g} : \mathbb{C}_m[x, y] \times \mathbb{C}_n[x, y] &\longrightarrow \mathbb{C}_{m+n}[x, y] \\ (q, p) &\longmapsto qf + pg \end{aligned}$$

where $f = \sum_{i=0}^{n} a_i(x)y^i$ has degree n and $g = \sum_{j=0}^{m} b_j(x)y^i$ has degree m in y, with respect to the chosen basis.

This homomorphism is given as follows for f degree 5 and g degree 3:

$$\chi_{f,g}(q,p) =$$

$$\begin{bmatrix} q_0(x) \\ q_1(x) \\ q_2(x) \\ p_0(x) \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ p_4(x) \end{bmatrix}^T \begin{bmatrix} a_0(x) & a_1(x) & a_2(x) & a_3(x) & a_4(x) & a_5(x) & 0 & 0 \\ 0 & a_0(x) & a_1(x) & a_2(x) & a_3(x) & a_4(x) & a_5(x) & 0 \\ 0 & 0 & a_0(x) & a_1(x) & {}_2(x) & a_3(x) & a_4(x) & a_5(x) \\ b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 & 0 & 0 & 0 \\ 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 & 0 & 0 \\ 0 & 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 & 0 \\ 0 & 0 & 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) & 0 \\ 0 & 0 & 0 & 0 & b_0(x) & b_1(x) & b_2(x) & b_3(x) \end{bmatrix} \begin{bmatrix} 1 \\ y \\ y^2 \\ y^3 \\ y^4 \\ y^5 \\ y^6 \\ y^7 \end{bmatrix}$$

$$= qf + pg$$

Proposition 2.2.13. The image of the $\mathbb{C}[x]$ module homomorphism $\chi_{f,g}$ is

$$\langle f, g \rangle \cap \mathbb{C}_{m+n}[x, y]$$

Proof. Clearly, $\text{im}(\chi) \subseteq \langle f, g \rangle$, and thus $\text{im}(\chi) \subset \langle f, g \rangle \cap \mathbb{C}_{m+n}[x, y]$.

To see $\langle f, g \rangle \cap \mathbb{C}_{m+n}[x, y] \subseteq \text{im}(\chi)$, take $q \in \mathbb{C}_m[x, y]$ and $p \in \mathbb{C}_n[x, y]$, then $\deg(qf) \leq m + n$ and $\deg(pg) \leq n+m$. Thus $\deg(qf+pg) \leq m+n$ and $qf+pg \in \mathbb{C}_{m+n}[x, y]$, but $qf+pg \in \langle f, g \rangle$ as well. Hence

$$\chi(q, p) = qf + pg \in \mathbb{C}_{m+n}[x, y] \cap \langle f, g \rangle,$$

and $\text{im}(\chi_{f,g}) = \langle f, g \rangle \cap \mathbb{C}_{m+n}[x, y]$. □

We will now investigate the relationship between the intersection multiplicity and intersection number of two curves

$$C = \zeta(f)$$

and

$$C' = \zeta(D')$$

without a common component at a point $p = (\alpha, \beta)$. To simplify matters, we will assume that the leading coefficients a_n of f and b_m of g are relatively prime.

We begin by showing that

(28) $$\mathbb{C}[x,y]\Big/\langle f,g\rangle \cong \mathbb{C}_{n+m}[x,y]\Big/\langle f,g\rangle\cap\mathbb{C}_{n+m}[x,y].$$

To do so, we show that for any $d > m+n-1$

$$\mathbb{C}_d[x,y]\Big/\langle f,g\rangle\cap\mathbb{C}_d[x,y] \cong \mathbb{C}_{d-1}[x,y]\Big/\langle f,g\rangle\cap\mathbb{C}_{d-1}[x,y].$$

As the leading coefficients of f and g are relatively prime, there are polynomials $\alpha(x)$ and $\beta(x)$ such that

(29) $$\alpha(x)a_n(x)+\beta(x)b_m(x)=1$$

Hence

$$\begin{aligned}
&\alpha(x)f(x,y)y^{d-n}+\beta(x)g(x,y)y^{d-m}\\
=&\sum_{i=0}^{n}\alpha(x)a_i(x)y^{i+d-n}+\sum_{i=1}^{m}\beta(x)b_i(x)y^{i+d-m}\\
=&\left(\alpha(x)a_n(x)+\beta(x)b_m(x)\right)y^d+\sum_{i=0}^{n-1}\alpha(x)a_i(x)y^{i+d-n}+\sum_{i=1}^{m-1}\beta(x)b_i(x)y^{i+d-m}\\
=&y^d+\sum_{i=0}^{n-1}\alpha(x)a_i(x)y^{i+d-n}+\sum_{i=1}^{m-1}\beta(x)b_i(x)y^{i+d-m}
\end{aligned}$$

Next, we define

$$h(x,y) := \sum_{i=0}^{n-1}\alpha(x)a_i(x)y^{i+d-n}+\sum_{i=1}^{m-1}\beta(x)b_i(x)y^{i+d-m}$$

and then given $d > m+n-1$, there is some $h(x,y)\in\mathbb{C}_d[x,y]$ such that

$$\alpha(x)f(x,y)y^{d-n}+\beta(x)g(x,y)y^{d-m}=y^d+h(x,y)$$

Hence,

$$\mathbb{C}_d[x,y]\Big/\langle f,g\rangle\cap\mathbb{C}_d[x,y] = \mathbb{C}_{d-1}[x,y]\Big/\langle f,g\rangle\cap\mathbb{C}_{d-1}[x,y]$$

and

$$\mathbb{C}[x,y]\Big/\langle f,g\rangle = \mathbb{C}_{n+m}[x,y]\Big/\langle f,g\rangle \cap \mathbb{C}_{n+m}[x,y]$$

So we have the following corollary:

Corollary 2.2.14. As a $\mathbb{C}[x]$ module, the quotient ring $\mathbb{C}[x,y]\Big/\langle f,g\rangle$ is isomorphic to the cokernel of the map $\chi_{f,g}$ defined by the Sylvester matrix.

Recall

$$\text{cokernel}(\chi_{f,g}) = \text{codom}(\chi_{f,g})\Big/\text{im}(\chi_{f,g})$$

Recall the Smith Normal Form of a matrix: as $\mathbb{C}[x]$ is a principal ideal domain, for any matrix A over $\mathbb{C}[x]$ we can find invertible matrices S and T with determinant 1, such that SAT is diagonal.

Viewing A, S, T and SAT as module homomorphisms, S and T are isomorphisms, and hence they induce an isomorphism between the cokernel of A and the cokernel of SAT. If the diagonal entries of SAT are $q_1(x), \ldots, q_d(x)$, then the cokernel of SAT is

$$\mathbb{C}[x]\Big/q_i(x)\mathbb{C}[x] \oplus \ldots \oplus \mathbb{C}[x]\Big/q_d(x)\mathbb{C}[x]$$

and hence

$$\text{dim}(\text{cokernel}(SAT)) = \sum_{i=1}^{d} \text{deg}(q_i)$$

which is also the degree of the product of the q_i's, or the degree determinant of SAT. As

$$\det(S) = \det(T) = 1$$

this is equal to the determinant of A. Thus the dimension of the cokernel of A is equal to the degree of det(A).

Applying the above to the Sylvester matrix, we know there are polynomials q_i such that

$$R_{f,g}(x) = \prod_{i=1}^{m+n} q_i(x)$$

where n is the y degree of f and m is the y degree of g and then

$$\mathbb{C}[x,y]\Big/\langle f,g\rangle \cong \text{coker}(\chi_{f,g}) \cong \bigoplus_{i=1}^{m+n} \mathbb{C}[x,y]\Big/q_i^{n_i}\mathbb{C}[x],$$

Thus the sum of the multiplicities of the common zeroes of f and g is equal to the degree of the resultant.

We can also consider $\mathbb{C}[x,y] / \langle f, g\rangle$ as a $\mathbb{C}[x]$ module, and localise at a point. The multiplicity of each root a of the resultant is equal to the sum of the multiplicities of the intersection points of f and g whose x coordinate is a.

Recall that as a $\mathbb{C}[x]$ module, $\mathbb{C}[x,y] / \langle f, g\rangle$ is the direct sum of modules $\mathbb{C}[x,y] / q_i\mathbb{C}[x]$, and so the localisation of $\mathbb{C}[x,y] / \langle f, g\rangle$ at a point $x = a$ is the direct sum of the localisations of $\mathbb{C}[x,y] / q_i\mathbb{C}[x]$ at $x = a$.

Corollary 2.2.15. With the notation from Lemma 2.2.12, the dimension of the localisation of $\mathbb{C}[x]$ module

$$\mathbb{C}[x] / p(x)\mathbb{C}[x]$$

at $x = a$ is the multiplicity of a as a root of $p(x)$.

Applying the above to the Sylvester matrix and $\chi_{f,g}$ we have that the dimension of the localisation of $\mathbb{C}[x,y] / \langle f, g\rangle$ (as a $\mathbb{C}[x]$ module) at the point a is the multiplicity of $a = (a_1, a_2)$ as a root of $R_{f,g}(x)$, if there is only one common zero of f and g with $x = a_1$.

The above discussion leads to the following Corollary:

Corollary 2.2.16. The dimension of $\left(\mathbb{C}[x,y] / \langle f, g\rangle\right)_{(a,b)}$ is equal to the multiplicity of (a, b) as a root of $R_{f,g}(x)$, where our coordinates are such that distinct common zeroes have distinct x and y values.

Example 7. Take the curves $C = \zeta(f(x, y) = x^2 - y)$ and $D = \zeta(g(x, y) = -x^2 - y)$. Recall that

$$R_{f,g} = -2x^2$$

As 0 is a double root of $-2x^2$, the intersection multiplicity of f and g at $(0, 0)$ is 2.

Now, consider ideals generated $\langle f, g\rangle$ and $\langle x, y\rangle$. Then

$$\langle f, g\rangle + \langle x, y\rangle \subseteq \langle f, g\rangle \subseteq \langle f, g\rangle + \langle x, y\rangle^2 =: J$$

Next, we find $\left(\mathbb{C}[x,y] / J\right)_{(0,0)}$ and determine its dimension.

Note $\mathbb{C}[x,y]\big/_J$ is the set of equivalence classes of polynomials with $q \equiv p$ if and only if $p - q$ is in J. This has dimension 2 as a vector space over $\mathbb{C}$ with basis given by the equivalence classes $[1]$ and $[x]$.

The localisation is

$$\left(\mathbb{C}[x,y]\big/_J\right)_{(0,0)} = \left\{\frac{[p]}{[q]} \mid [p],[q] \in \mathbb{C}[x,y]\big/_J [q], \neq [0]\right\}$$

which has dimension 2.

Having seen that the intersection number and intersection multiplicity are equal, we discuss the generic type of intersection between two curves: when $\nu_p(C,D) = 1$.

Definition 2.2.17. Two irreducible curves C and D in $\mathbb{C}^2$ intersect transversally at p if

$$\nu_p(C,D) = 1.$$

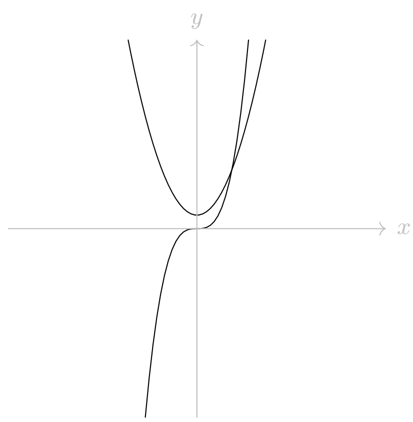


FIGURE 4. Transversal intersection

Two curves C and D intersect transversally at a point p if and only if the following three conditions are satisfied:

(a) p must be a regular point of f

(b) p must be a regular point of g

(c) the tangent spaces of f and g at p must be distinct

This means that C and C' intersect transversally at p if and only if the tangent space of $\mathbb{C}^2$ at p is the direct sum of the tangent spaces of C and C' at p.

2.3. Revisiting the Multiplicity of a Point

Having seen that we can describe the intersection of two curves at a point independent of choice of coordinates, we briefly return the multiplicity of a curve C at a point p, and provide an analogous coordinate independent definition. For this section, we consider an irreducible curve in $\mathbb{C}^2$.

Following [2], we see that Definition 1.0.2 is equivalent to considering the dimension of the quotient of large enough powers of the maximal ideal of the variety at the point p (Definition 2.2.11).

As discussed, the local ring is the set of rational functions on the variety C which are defined at p, and $\mathfrak{m}_p(C)$ is the set of rational functions on C which are 0 at p. The connection between the intersection multiplicity and intersection number of two curves C and D at a point p was that the degree of p as a root of the resultant is equal to the dimension of the localisation of the quotient ring $\mathbb{C}[x,y] / \langle f, g\rangle$, where $C = \zeta(f)$ and $D = \zeta(g)$. The corresponding relationship for the multiplicity of C at p is the following:

Theorem 2.3.18. Let p be a point on the irreducible curve $C = \zeta(f)$, then for $n \gg 0$

$$\nu_p(C) = \dim_{\mathbb{C}} \left(\mathfrak{m}_p(C)^n / \mathfrak{m}_p(C)^{n+1} \right)$$

Proof. Consider the following exact sequence:

$$0 \longrightarrow \mathfrak{m}_p(C)^n / \mathfrak{m}_p(C)^{n+1} \longrightarrow \mathcal{O}_p(C) / \mathfrak{m}_p(C)^{n+1} \longrightarrow \mathcal{O}_p(C) / \mathfrak{m}_p(C)^n \longrightarrow 0 \tag{210}$$

so it is sufficient to show

$$\dim_{\mathbb{C}} \left(\mathcal{O}_p(C) / \mathfrak{m}_p(C)^n \right) = n\nu_p(C) + s$$

with s some constant and $n \geq \nu_p(C)$. Without loss of generality, we may assume that $p = (0,0)$, and hence

$$\mathfrak{m}_p(C) = \langle x, y\rangle$$

Note that $\zeta(\langle x, y\rangle^n) = \{(0,0)\}$, and thus

$$\mathbb{C}[x,y] / \langle \mathfrak{m}_p(C)^n, f\rangle \mathcal{O}_p(\mathbb{C}^2) \cong \mathcal{O}_p(C) / \langle x, y\rangle^n \mathcal{O}_p(C) \cong \mathcal{O}_p(C) / \mathfrak{m}_p(C)^n$$

We have reduced the situation to calculating $\dim_{\mathbb{C}} \left(\mathbb{C}[x,y] \big/ \langle \mathfrak{m}_p(C)^n, f \rangle \right)$. We can construct another exact sequence:

$$0 \longrightarrow \mathbb{C}[x,y] \big/ \mathfrak{m}_p(C)^{n-\nu_p(C)} \longrightarrow \mathbb{C}[x,y] \big/ \mathfrak{m}_p(C)^n \longrightarrow \mathbb{C}[x,y] \big/ \langle \mathfrak{m}_p(C)^n, C \rangle \longrightarrow 0$$

And thus for all $n \geq \nu_p(C)$,

$$\dim_{\mathbb{C}} \left(\mathbb{C}[x,y] \big/ \langle \mathfrak{m}_p(C)^n, C \rangle \right) = n\nu_p(C) - \frac{\nu_p(C)(\nu_p(C) - 2)}{2}$$

Returning to (210), we have

$$\begin{aligned} \dim_{\mathbb{C}} \left(\mathfrak{m}_p(C)^n \big/ \mathfrak{m}_p(C)^{n+1} \right) &= \dim_{\mathbb{C}} \left(\mathcal{O}_p(C) \big/ \mathfrak{m}_p(C)^{n+1} \right) - \dim_{\mathbb{C}} \left(\mathcal{O}_p(C) \big/ \mathfrak{m}_p(C)^n \right) \\ &= \frac{(n+1)\nu_p(C) - \nu_p(C)(\nu_p(C) - 2)}{2} - \frac{n\nu_p(C) - \nu_p(C)(\nu_p(C) - 2)}{2} \\ &= \nu_p(C) \end{aligned}$$

Hence

$$\dim_{\mathbb{C}} \left(\mathfrak{m}_p(C)^n \big/ \mathfrak{m}_p(C)^{n+1} \right) = \nu_p(C)$$

□

2.4. Puiseux Expansions

In the Section 2.1, we eliminated powers of y from two simultaneous equations: we sought an equation which x-values of common zeroes must satisfy. The zeroes of the resultant of $f(x,y)$ and $g(x,y)$ were candidate x-values for common zeroes. All that remains is find corresponding y-values so that (x,y) is a common zero of $f(x,y)$ and $g(x,y)$, which we do by solving a polynomial in y. Doing so, however, is not trivial: given polynomials of degree greater than five in y, there is no general method for finding the roots. If however, $f(x,y) = 0$ and $g(x,y) = 0$ satisfy the conditions of the Holomorphic Implicit Function Theorem, then there exists an expression of y as a convergent power series in x. Unfortunately, it is not true that either f and g will satisfy these conditions in general: consider the polynomial $f(x,y) = y^p - x^q$, with $q > p$, given an x, $y^p = x^q$, that is $y = x^{\frac{q}{p}}$, and so to find the corresponding y, we must allow fractional powers of x. To obtain such a fractional power series, a *Puiseux expansion* of a polynomial f, we introduce the *Newton polygon* of f to decompose f into components for which we can find a factional power series expression for y in terms of x. We use the Newton polygon and certain exponents in the Puiseux expansion in Section 3.1 to show that we a finite sequence of σ-processes will resolve a singularity. We follow the discussion in [1].

We generalise this by finding polynomials have solutions of the form

$$y = tx^{\mu}$$

with $t \in \mathbb{C}$ and $\mu = \frac{q}{p} \in \mathbb{Q}_{\geq 0}$. These are polynomials $f(x, y)$ such that

$$f(x, tx^{\mu}) = \sum a_{ij} t^j x^{i+j\mu} = 0.$$

Recall a homogenous polynomial is a polynomial $f(x_1, \ldots, x_n)$ such that

$$f(tx_1, \ldots, tx_n) = t^n f(x_1, \ldots, x_n),$$

and a quasi-homogenous polynomial is one such that there are weights ω_i with

$$f(t^{\omega_i} x_1, \ldots, t^{\omega_n} x_n) = t^n f(x_1, \ldots, x_n).$$

If f is quasi-homogeneous, then

$$\begin{aligned} f(x, tx^{\mu}) &= \sum_{i+\mu j=\nu} a_{ij} x^i t^j x^{j\mu} \\ &= \sum_{i+\mu j=\nu} a_{ij} x^{i+j\mu} t^j \\ &= x^{\nu} \sum a_{ij} t^j \\ &= x^{\nu} g(t) \end{aligned}$$

and so letting t_0 be a zero of $g(t)$, $y = t_0 x^{\mu}$ is a solution of $f(x, y) = 0$. We can assume that $t_0 \neq 0$ as long as $g(t) \neq ct^m$, which occurs when $f(x, y)$ consists of at least two distinct monomials.

We can also interpret the assumption that $f(x, y)$ is quasi-homogeneous geometrically. Note that each monomial $x^i y^j$ corresponds to an element (i, j) of the lattice $\mathbb{N}^2$, and so given a general polynomial

$$f(x, y) = \sum a_{ij} x^i y^j,$$

we consider the set $\Delta(f) = \{(i, j) \in \mathbb{N}^2 \mid a_{ij} \neq 0\}$, called the carrier of f.

Now, if $f(x, y)$ is quasi-homogenous, then there are rational numbers μ and ν such that $i + \mu j = \mu$ for all $(i, j) \in \Delta(f)$. This means that all the points $\Delta(f)$ lie on the line $i + \mu j = \nu$ in $\mathbb{R}^2$, and this line has slope $-\frac{1}{\mu}$ and intersections the i-axis at $i = \nu$.

We provide an example of this process.

Example 8. Take $f(x,y) = x^3 + x^2y + 4y^3 = 0$.

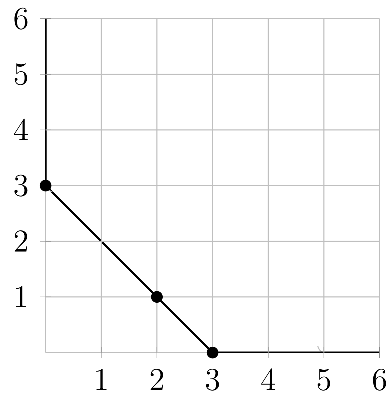


FIGURE 5. Newton Polygon of $f(x,y) = x^3x^2y + 4y^3 = 0$.

In this case, $\mu = 1$, $\nu = 3$, and so we have

$$f(x,tx) = x^3(1 + t + 4t^3),$$

with $g(t) = 1+t+t^3$. The roots of g are $t = \frac{-1}{2},\ \frac{1}{4}(1-i\sqrt{7}),\ \frac{1}{4}(1+i\sqrt{7})$. And so our solutions are $y = \frac{-1}{2}x,\ \frac{1}{4}(1-i\sqrt{7})x,\ \frac{1}{4}(1+i\sqrt{7})x$.

As we know how to find solutions y of $f(x,y)$ in terms of powers of x when f is quasi-homogeneous, we look for a method of partitioning any polynomial $f(x,y) \in \mathbb{C}[x,y]$ into a quasi-homogeneous component and a remainder.

To form such a partition, we construct a convex hull containing $\Delta(f)$, and consider its boundary. This convex hull is the intersection of all half planes which contain all the points of $\Delta(f)$ and do not contain a point of the third quadrant. The boundary of this convex hull consists of a compact polygonal path and two half-lines. We call this compact polygonal path the *Newton polygon* of f, denoted $\mathfrak{N}(f)$. We will use the components of the Newton polygon to obtain our desired partition of $f(x,y)$.

We begin by finding a power series expression for the lower terms of $f(x,y)$, and these correspond to the steepest segment of $\mathfrak{N}(f)$. Let the slope of the steepest segment of $\mathfrak{N}(f)$ be $-\frac{1}{\mu_0}$ and the continuation intersects the i-axis at ν_0. We now partition $f(x,y)$ in to a quasi-homogeneous component $\tilde{f}(x,y)$ and a remainder $r(x,y)$ using μ and ν as follows:

$$f(x,y) = \sum_{i+\mu_0 j=\nu_0} \alpha_{ij}x^iy^j + \sum_{i+\mu_0 j>\nu_0} \alpha_{ij}x^iy^j. \tag{211}$$

We know how to find solutions y in terms of x for the quasi-homogeneous component as $y = t_0 x^{\mu_0}$, where t_0 is a zero of $g(t)$ as above. Thus, $y = t_0 x^{\mu}$ is a good approximation for solutions y of $f(x,y)$.

In fact, the error of this approximation is precisely

$$r(x, t_0^{\mu_0}) = \sum_{i+\mu_0 j > \nu_0} \alpha_{ij} x^{i+\mu_0 j} t_0^j,$$

and so to obtain a better approximation of solutions y to $f(x,y)$, we substitute $y = x_1^q(t_0 + y_1)$, with $\mu_0 = \frac{q_0}{p_0}$, into $f(x,y)$:

$$f(x_1^{q_0}, x_1^{q_0}(t_0 + y_1)) = x_1^{\nu_0 p_0} f_1(x,y)$$

as we know $x_1^{\nu p}$ divides $f(x_1^q, x_1^q(t_0 + y_1))$ by (211).

Next, we consider the carrier of $f_1(x,y)$, and repeat the process, we obtaining an expression:

$$y = x^{\mu_0} \left(t_0 x_1^{\mu_1} \left(t_1 + x^{\mu_2} \left(+ \ldots \right) \right) \right)$$

which is an expansion of y as a series of increasing fractional powers of x. For such an expression to be meaningful, the denominators of the exponents should eventually stabilise, that is, there should be an n such that the largest denominator is n.

To see such an n exists, let m_i be the y-generality order of f_i obtained above. If $m_{i+1} = m_i$, then $g_{i+1}(t) = c(t - t_0)^{m_i}$ and the coefficient $\alpha_{a,m-1}$ of t^{m-1} does not vanish. In this case, $a + \mu_i(m-1) = \mu_i m$, and $\mu_i \in \mathbb{N}$. The m_i form a descending sequence in $\mathbb{N}$, hence there are only finitely many instances where $m_{i+1} \neq m_i$. Thus, we can find an index j such that for all $i \geq j$ $m_{i+1} = m_i$. Letting n be the least common multiple of the m_i for $i \leq j$, we can express y as a power series in $x^{\frac{1}{n}}$.

Thus, the following definition makes sense.

Definition 2.4.19. Given a polynomial $f(x,y) = \sum \alpha_{ij} x^i y^j = 0$, the expression $y = \sum a_i x^{\frac{i}{n}}$, as defined above (the powers series in $x^{\frac{1}{n}}$) is called the *Puiseux (Series) expansion* of f.

Example 9. Take $f(x,y) = 3x^5 + x^2 y + 4y^3 = 0$.

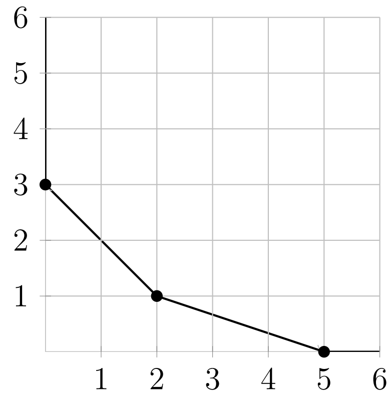


FIGURE 6. Newton Polygon of $f(x,y) = 3x^5 + x^2y + xy^2 + 4y^3 = 0$.

In this case, $\mu_0 = 1$ and $\nu_0 = 3$, and so we form the partitions:

$$f(x,y) = (x^2y + 4y^3) + (3x^5)$$

As a first approximation, we find a solution y for the quasi-homogeneous part

$$\overline{f}(x,y) = x^2y + 4y^3$$

by substituting $y = tx$, we have

$$\overline{f}(x,t) = x^3(t + 4t^3).$$

Now, $t_0 = -\frac{i}{2}$ is a non-zero root of $g(t)$, and so

$$y_0 = -\frac{i}{2}x$$

is the first approximate of the solution of $f(x,y) = 0$.

We improve our approximation by substituting $y = x(-\frac{i}{2} + y_1)$ into $f(x,y) = 0$, obtaining a new power series in x and y_1:

$$f(x,y_1) = 3x^5 + x^3(-\frac{i}{2} + y_1) + x^3(-\frac{i}{2} + y_1)^2 + 4x^3(-\frac{i}{2} + y_1)^3$$
$$f(x,y_1) = x^3(5x^2 + (-\frac{i}{2} + y_1) + y_1^2 - iy_1 - \frac{1}{4} + y_1^3 - \frac{3iy_1^2}{2} - \frac{3y_1}{4} + \frac{i}{8})$$

and repeating the process with $f_1(x,y_1) = 5x^2 + (-\frac{i}{2} + y_1) + y_1^2 - iy_1 - \frac{1}{4} + y_1^3 - \frac{3iy_1^2}{2} - \frac{3y_1}{4} + \frac{i}{8}$.

Assume $f(x,y)$ is irreducible, y-general of order m and has multiplicity m at $(0,0)$, then the Puiseux expansion of f is of the form

$$y = a_{k_1}x^{k_1} + \sum a_i x^i \quad k_1 \geq 1, k_i > k_{i-1}, k_1 \in \mathfrak{Q}$$

which we can also express parametrically as follows:

$$x = t^m$$
$$y = a_1 t^{k_1} + a_2 t^{k_2} + \dots$$

which we call the *parametric Puiseux expansion* of $f(x,y) = 0$.

In Chapter 4, we examine the parametric Puiseux expansion of a polynomial f, to show that ways in which we represent a standard resolution of a curve $C = \zeta(f)$ are equivalent. To do so, we examine the relationship between the multiplicity of the point under each σ-process, and relate them to the *characteristic Puiseux exponents of f* [3].

Definition 2.4.20. Let $f(x,y) = \sum \alpha_{ij}x^i y^j = 0$ be irreducible, with Puiseux parametric expansion $x = t^m, y = \sum a_i x^{k_i}$, where $m \leq k_1 < k_2 < \ldots \in \mathbb{Z}$ and $a_i \neq 0$ for all i. We define the *Puiseux characteristic exponents of f* as follows:

Set $\Gamma_0 = m$ and $\Gamma_j = \gcd(m, k_1, k_2, \ldots, k_j)$ for $j \geq 1$.

The Γ_i form a non-increasing sequence of positive integers, which stabilises for some j_0 with $\Gamma_j = 1$ for all $j \geq j_0$. We now define the sequence of exponents $\Lambda_0 < \ldots \Lambda_l$, setting $\Lambda = m$ and defining the other Λ_j's as the k_j such that $\Gamma_{j-1} > \Gamma_j$. These Λ_i are the *characteristic Puiseux exponents*.

In Section 3.1, we examine the Newton polygons of curves under σ-processes to investigate how the intersection between the *strict pre-images* of two curves C and D are related to the degree to which D *contacts* C. This requires the following results about Newton polygons, collected from [1].

Lemma 2.4.21. Take a curve $C = \zeta(f(x,y)) \subset \mathbb{C}^2$

(a) if C is irreducible and different from the coordinate axes, the Newton polygon of f consists of a single segment.

(b) if C is a reducible curve, with irreducible components C_i, the segments of the Newton polygon of C are the Newton polygons of the C_i suitably shifted. That is, take the irreducible components of C, say C_l, with $\mathfrak{N}(C_l)$ the segment between $(0, p_l)$ and $(q_l, 0)$. Then $\mathfrak{N}(C)$ consists of the segments ℓ_l, where ℓ_l is between

$$\left(\sum_{k<l} q_k, \sum_{j \geq k} p_j \right)$$

and

$$\left(\sum_{k \leq l} q_k, \sum_{j > k} p_j \right).$$

We will use this Lemma to examine the affect of σ-processes on the Newton polygon of a curve, and show that after a finite sequence of σ-processes, the strict transform C' is regular.

CHAPTER 3

Resolving Singularities

> Perhaps you have been looking in the wrong places.
>
> *J.K. Rowling*
> *Harry Potter and The Half-Blood Prince*

This chapter studies resolving singularities of a plane curve by means of σ-processes. We *blow up* the singular point p of the curve $C \subset \mathbb{C}^2$ and project back to obtain a *birationally equivalent* plane curve C', whose points corresponding to $p \in C$ have lower multiplicity than p. We show that after a finite number of σ-processes we arrive at a regular curve. Using the Newton polygons of the curve C and a curve D which has *maximal contact* with C at the singularity, we obtain an upper bound for the number of iterations required. We introduce an algorithm to increase the efficiency of this process when the singularity is an ordinary singularity. Quadratic transformations allow us to reduce non-ordinary singularities to ordinary ones. Since σ-processes are local and singularities are isolated, resolving one singularity does not affect the others, allowing us to resolve singularities simultaneously, as demonstrated in Section 3.4. Initially, we assume that the singular curve C is *irreducible*, that is, it is not the union of two curves $C^{(1)}$ and $C^{(2)}$. The final Section shows how to extend our results to curves which are not necessarily irreducible.

We will now formulate the notion of *birational equivalence* between two curves. We begin defining what a *morphism* from C to D is. Take a map $\varphi : C \longrightarrow D$. If φ is continuous, and for any open subset U of D and a *rational polynomial* f defined on U, the function $f \circ \varphi$ is a rational polynomial defined on $\varphi^{-1}(U)$ in C, then we call φ a *morphism* from C to D. A rational polynomial on an open subset U of D is the quotient of two polynomial functions $f(x, y), g(x, y) \in \mathbb{C}[x, y]$ from U to $\mathbb{C}$, such that $g(x, y) \neq 0$ for all $(x, y) \in U$.

We can now define *birational equivalence* [2].

Definition 3.0.1. Let C and D be two curves of $\mathbb{CP}^2$, and U_1, U_2 open subsets of C two morphisms $f_1 : U_1 \longrightarrow D$ and $f_2 : U_2 \longrightarrow D$ are equivalent if $f_1 |_{U_1 \cap U_2} = f_2 |_{U_1 \cap U_2}$. The equivalence class $\left[f_1\right]$, denoted $\mathcal{F}$, of such morphisms is called a *rational map* from X to Y. The rational map $\mathcal{F}$ is called *birational* if there are open subsets $U \subset C$ and $V \subset D$ and a representative f

of $\mathcal{F}$, such that $f : U \longrightarrow V$ is an isomorphism. In this case, we say C and D are birationally equivalent.

Remark 3.0.2. In general, the notion of birationally equivalence allows us to ignore *subvarities* of lower dimension. For a curve C, this amounts to removing a finite set of points, say A. Birational maps also respect the local rings at points in $C \setminus A$.

3.1. σ-processes

Recall that we moved from considering curves in $\mathbb{C}^2$ to considering them in $\mathbb{CP}^2$, as this allowed us to define an invariant of the intersection of two curves. This invariant was the sum of the intersection multiplicities, which is equal to the product of the degrees of the defining polynomials. This move also allowed us to see singularities of a curve in $\mathbb{C}^2$ which occur at *infinity*: for example, $C = \zeta(x^2y - 1)$ is a non-singular curve, but homogenising, we obtain $X^2Y - Z^3$, which has a singularity at the point $[0 : 1 : 0]$ in $\mathbb{CP}^2$.

While we have moved to considering curves in $\mathbb{CP}^2$, blowing up a point $p \in \mathbb{CP}^2$ is a local process: it does not affect points in other coordinate patches of $\mathbb{CP}^2$. Thus, when resolving singularities of curves, we blow up at a point p in $\mathbb{C}^2$. By a coordinate transformation, we may assume that $p = (0, 0)$, and thus only define a blow up at $(0, 0)$ [4].

Definition 3.1.3. Let $\mathcal{B} = \{((x, y), [u : v]) \mid xv = yu\} \subset \mathbb{C}^2 \times \mathbb{CP}^1$, and

$$\pi : B \longrightarrow \mathbb{C}^2$$
$$((x, y), [u : v]) \longmapsto (x, y)$$

Then $\pi^{-1}(\mathbb{C}^2)$ is a blow up of $\mathbb{C}^2$ at $(0, 0)$.

Given a curve $C \subset \mathbb{C}^2$, the *strict pre-image* of C is

$$\widetilde{C} := \overline{\pi^{-1}(C \setminus \{(0, 0)\}},$$

and the *exceptional line* is

$$E := \pi^{-1}\left((0, 0)\right).$$

Note that:

(i) $\pi^{-1}\left((0, 0)\right) = \{(0, 0)\} \times \mathbb{CP}^1$

(ii) $\pi|_{B \setminus \left\{\{(0,0)\} \times \mathbb{CP}^1\right\}} : B \setminus \left\{\{(0, 0)\} \times \mathbb{CP}^1\right\} \longrightarrow \mathbb{C}^2 \setminus \{(0, 0)\}$ is an isomorphism.

Remark 3.1.4. We can think of the space $\mathcal{B}$ attaching to each point (x, y) information about the slope of the secant at (x, y), which is $\frac{y}{x}$. The slope of the secant is the *height* of the point $(x, y, \frac{y}{x})$ in $\mathcal{B}$. The only point at which this analogy fails is $(0, 0)$. The pre-image of $(0, 0)$ we called the exceptional line, a copy of $\mathbb{CP}^1$.

$\mathbb{CP}^1$ is the union of the coordinate charts,M_1 and M_2, where

$$M_1 = \{[1 : v] | v \in \mathbb{C}\}$$

and

$$M_2 = \{[u : 1] | u \in \mathbb{C}\},$$

each of which of is homeomorphic to $\mathbb{C}$. This allows us to identify $B \cap (\mathbb{C}^2 \times M_i)$ with $\mathbb{C}^2$ by means of the maps

$$\begin{aligned} \widetilde{\pi}_1 : B \cap (\mathbb{C}^2 \times M_1) &\longrightarrow \mathbb{C}^2 \\ ((x, xv), [1 : v]) &\longrightarrow (x, v) \\ \widetilde{\pi}_2 : B \cap (\mathbb{C}^2 \times M_2) &\longrightarrow \mathbb{C}^2 \\ ((yu, y), [u : 1]) &\longrightarrow (u, y). \end{aligned}$$

The strict pre-image of a curve $C \subset \mathbb{C}^2$ is no longer a plane curve, however, $C' := \widetilde{\pi}_i(\widetilde{C})$ is another plane curve, the *strict transform* of C.

Remark 3.1.5. We still refer to $\widetilde{\pi}_i(E)$ as the *exceptional line*.

While $\widetilde{\pi}_i^{-1}$ is not a function, $\pi \circ \widetilde{\pi}_i^{-1}$ is.

Definition 3.1.6. Let

$$\Upsilon = \pi \circ \widetilde{\pi}_i^{-1},$$

then Υ^{-1} is a σ-process at $(0, 0) \in \mathbb{C}^2$.

Remark 3.1.7. We obtain two σ-processes

$$\Upsilon_1 = \pi \circ \widetilde{\pi}_1^{-1} : \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid x = 0\} \longrightarrow \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid x = 0\},$$

and

$$\Upsilon_2 = \pi \circ \widetilde{\pi}_2^{-1} : \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid y = 0\} \longrightarrow \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid y = 0\},$$

For $(x, y) \in \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid x = 0\}$,

$$\Upsilon_1(x, y) = (x, xy),$$

and

$$\Upsilon_1^{-1}(x, y) = (x, \frac{y}{x}),$$

and if $(x, y) \in \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid y = 0\}$,

$$\Upsilon_2(x, y) = (xy, y),$$

and

$$\Upsilon_2^{-1}(x,y) = (\frac{x}{y}, y).$$

In either case we take

$$C' = \overline{\Upsilon^{-1}(C \setminus \{(0,0)\})}$$

as the strict transform of C.

If no tangent to the curve C at the singularity has slope 0 or $\frac{1}{0} = \infty$, then either Υ_1 or Υ_2 is. If the slope of a tangent is 0, we use Υ_1, whereas if the slope is ∞, then we use Υ_2. The corresponding coordinate charts on $\mathcal{B}$ *ignore* the issues presented by such gradients.

Remark 3.1.8. Traditionally, *blow ups* and σ-processes are used interchangeably to refer to Definition 3.1.3, in this thesis we distinguish the two: a blow up is the map from Definition 3.1.3, and it does not generate another plane curve, whilst a σ-process as in Definition 3.1.6 does.

Proposition 3.1.9. The map Υ is a birational map from C' to C.

Proof. Let $p = (0,0)$ be a singularity of C, and let $\mathcal{P} = \Upsilon^{-1}(p)$.

Then $\mathcal{P} \subset \{(x,y) \in \mathbb{C}^2 \mid x = 0\}$.

Now, Υ induces an isomorphism between

$$\mathbb{C}^2 \setminus \{(x,y) \in \mathbb{C}^2 \mid x = 0\}$$

and

$$\mathbb{C}^2 \setminus \{(x,y) \in \mathbb{C}^2 \mid x = 0\},$$

so induces an isomorphism between

$$C' \cap \left(\mathbb{C}^2 \setminus \{x = 0\}\right)$$

and

$$C \cap \left(\mathbb{C}^2 \setminus \{x = 0\}\right).$$

Hence Υ is a birational map from C' to C. □

So, given a $C \subset \mathbb{C}^2$, $C' = \overline{\Upsilon^{-1}(C \setminus \{(0,0)\})}$ is a birationally equivalent plane curve. We have a way of obtaining birationally equivalent curves: σ-processes. Unfortunately, a *single* application of a σ-processes to a singular curve does not necessarily generate a non-singular curve.

Example 10. Take $f(x, y) = y^2 - x^5 \in \mathbb{C}[x, y]$ and let C the zero set of $f(x, y)$.

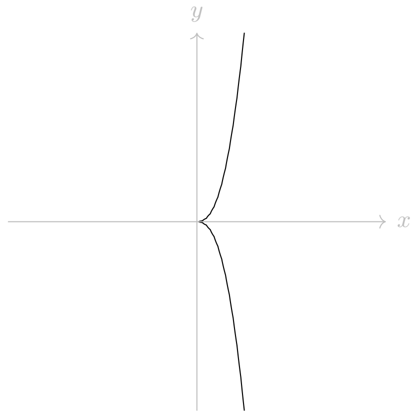


FIGURE 7. $C = \zeta(y^2 - x^5)$

To apply the σ-process, set $y = xy$: $f^{(1)} = x^2(y^2 - x^3)$, with E_1 defined by $x = 0$

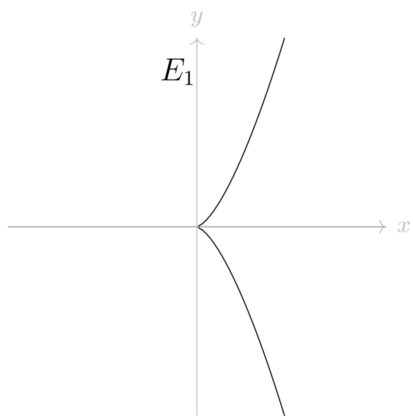


FIGURE 8. $\zeta(y^2 - x^3)$

In this case, $C' = \zeta(v^2 - x^3)$ is still a singular curve.

Thus, the application of a σ-process does not necessarily result in a non-singular curve. However, the complexity of the singularity has been reduced, and we can iterate applications of the σ-process. The singularity of a curve is *resolved* when a finite sequence of σ-processes produces a non-singular curve. We formalise this by defining a *standard resolution* of a singularity.

Definition 3.1.10. Let $\mathbb{C}^2_k \xrightarrow{\Upsilon_k} \mathbb{C}^2_{k-1} \xrightarrow{\Upsilon_{k-1}} \ldots \xrightarrow{\Upsilon_2} \mathbb{C}^2_1 \xrightarrow{\Upsilon_1} \mathbb{C}^2$ be a sequence of σ-process centred at the singularity $(0, 0)$ of the irreducible curve $C \subset \mathbb{C}^2$.

Then $C_i \subset \mathbb{C}^2_i$ is the i^{th} strict pre-image of C, and $E_i = \Upsilon_i(0,0)$ is the exceptional line introduced by the i^{th} σ-process.

We call $C_i \subset \mathbb{C}^2_i$ a *standard resolution* of the singularity $(0,0)$ of C if C_i satisfies the following conditions:

(a) for all $j \geq i$, $C_j \subset \mathbb{C}^2_j$ is non-singular

(b) $E_k \cap E_l \cap C_i = \emptyset$ for all $k \neq l$

(c) C_i intersects $E = \cup_{k=1}^{j} E_k$ transversally.

Remark 3.1.11. We impose (b) and (c) on C_i, as these ensure that we can decompose the pre-image of a singular curve C under Υ into the exceptional lines and the strict pre-image nicely. That is, the intersections between E and C_i should not have a multiplicity greater than 1.

3.1(a). Examples. We now provide some examples of resolving singularities of irreducible curves in $\mathbb{C}^2$.

Recall Remark 3.1.7

$$\Upsilon_1^{-1}(x,y) = (x, xy)$$

and

$$\Upsilon_2^{-1}(x,y) = (xy, y).$$

Example 11. Take $f(x,y) = y^2 - x^3 \in \mathbb{C}[x,y]$ and let C be the zero set of $f(x,y)$.

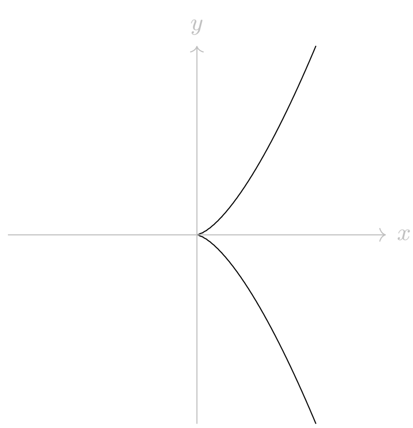


FIGURE 9. $\zeta(y^2 - x^3)$

Then C has a singularity $(0,0)$ with multiplicity 2.

As the tangent to C at $(0,0)$ has slope 0, we substitute $y = xy$ for the first σ-process, obtaining the polynomial $x^2y^2 - x^3 = x^2(y^2 - x)$, with $C_1 = \zeta(y^2 - x)$ and E_1 defined by $x = 0$

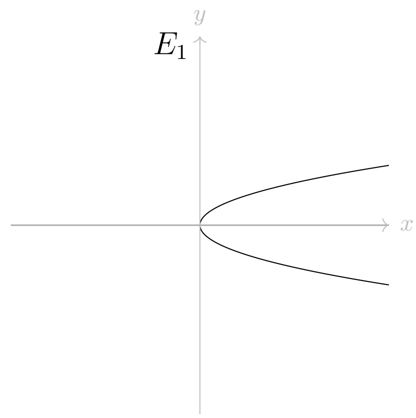


FIGURE 10. First strict transform of $(0,0) \in \zeta(y^2 - x^3)$.

The tangent to C_1 at $(0,0)$ has infinite slope, so we substitute $x = xy$: $x^2y^4 - x^3y^3 = y^3x^2(y-x)$, with $C_2 = \zeta(y-x)$ and E_2 defined by $y = 0$,

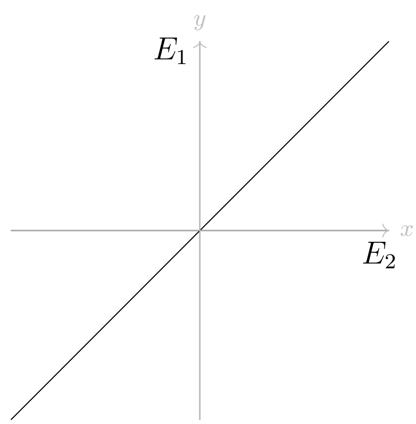


FIGURE 11. Second strict transform of $(0,0) \in \zeta(y^2 - x^3)$.

For the third σ-process, we substitute $y = xy$: obtaining $y^5x^2(y - xy) = y^6x^2(1-x)$, with $C_3 = \zeta(1-x)$ and exceptional lines $E_1 = E_3$ given by $x = 0$ and $E - 2$ given by $y = 0$, which is a standard resolution.

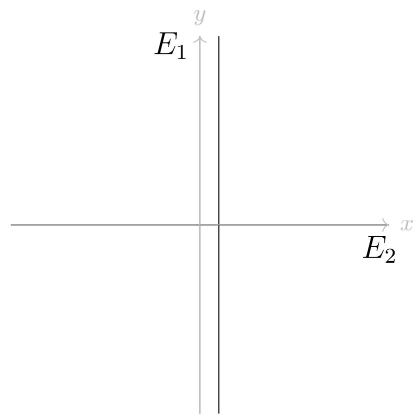


FIGURE 12. The standard resolution of the singularity $(0,0) \in \zeta(y^2 - x^3)$.

Example 12. Let $f(x,y) = y^2 - x^3 - x^2 \in \mathbb{C}[x,y]$ and let C be the zero set of $f(x,y)$.

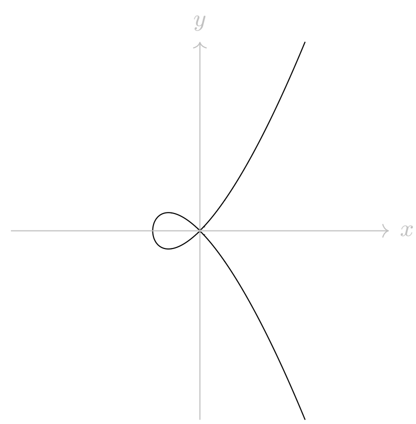


FIGURE 13. $\zeta(y^2 - x^3 - x^2)$

Then C has a singularity at $(0,0)$ with multiplicity 2.

For the σ-process, we substitute $y = xy$:, obtaining $x^2y^2 - x^3 - x^2 = x^2(y^2 - x - 1)$, with $C_1 = \zeta(y^2 - x - 1)$ and E_1 defined by $x = 0$, which is a standard resolution.

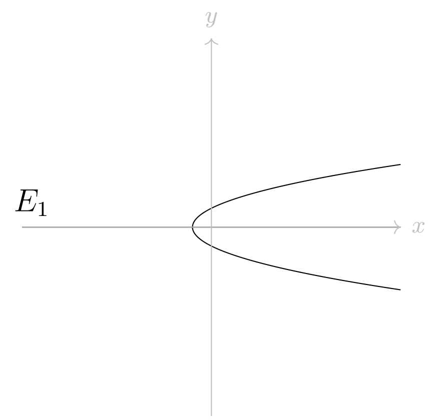


FIGURE 14. Standard resolution of singularity $(0,0) \in \zeta(y^2 - x^3 - x^2)$.

Example 13. Take $f(x,y) = y^2 - x^5 \in \mathbb{C}[x,y]$ and let C be the zero set of $f(x,y)$.

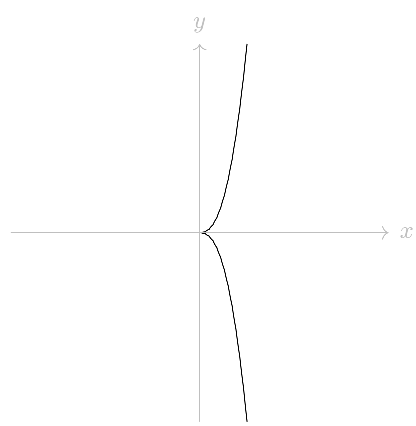


FIGURE 15. $C = \zeta(y^2 - x^5)$

For the first σ-process, we substitute $y = xy$, obtaining $x^2(y^2 - x^3)$, with $C_1 = \zeta(y^2 - x^3)$ and E_1 defined by $x = 0$.

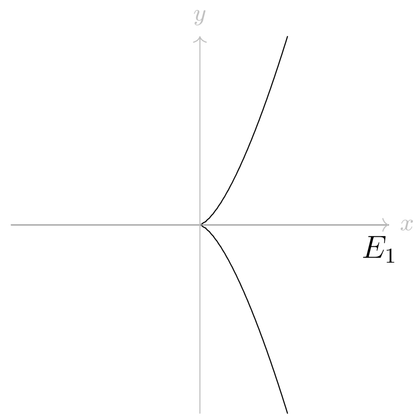


FIGURE 16. $\zeta(y^2 - x^3)$

We reach a standard resolution by blowing up the singularity $(0, 0)$ of $\zeta(y^2 - x^3)$, as in Example 11.

3.2. Existence of Standard Resolutions

To show that a standard resolution exists, we construct a smooth curve D through the singular point p of C which approximates C so well that the strict pre-images D_i of D remain tangential to the strict pre-images C_i of C at the points in the pre-image of p with the same multiplicity as p. After each σ-process, we will choose our coordinates such that the strict pre-image D_i and new exceptional line E_i are the axes, and consider the Newton polygon of C_i. We use the gradients of the segments of $\mathfrak{N}(C_i)$ to measure the amount by which we have resolved the singularity. We then show that a finite sequence of σ-processes reduces the multiplicity of p. This then allows us to show that after a finite sequence of σ-processes, we obtain a strict pre-image which is a non-singular plane curve.

3.2(a). Maximal Contact. To construct such a curve D, we introduce the *contact exponent* of C and D at a point p, and the *first contact exponent* of C at p, as well as the notion of curves having *maximal contact*. We then examine the conditions under which the x-axis has maximal contact with a curve at $(0, 0)$. After this, we show that the first contact exponent of a curve at a singularity is finite, before introducing *infinitely near* points p_i of a singularity p. We prove that the bound on the number of infinitely near points p_i with the same multiplicity $\nu_p(C)$ is the integer part of the first contact exponent of C at p.

We temporarily remove our assumption that the curves we consider are irreducible, allowing us to remove it completely in the last section of this chapter.

The intersection multiplicity $\nu_p(C, D)$ of C and D measures the contact between C and D at p, in fact, it counts the number of infinitely near points in of p which are common to C and D. It can be shown that for an irreducible curve C, D has maximal contact with C at p if $\nu_p(C, D)$ is the supremum of $\nu_p(C, \widetilde{D})$, over all smooth curve $\widetilde{D}$ through p [1].

If C is reducible, then we require more subtlety. Comparing intersection numbers $\nu_p(C^{(i)}, D)$ between irreducible components of $C^{(i)}$ of C, does not allow us to compare the contact of D with each $C^{(i)}$. Note that as D is smooth, $\nu_p(C^{(i)}, D) \geq \nu_p(C_i)$, and equality holds if the intersection is transversal, and it is possible that $\nu_p(C^{(i)}, D) > \nu_p(C^{(i)}, D)$ as $\nu_p(C^{(i)}) > \nu_P(C^{(j)})$, even if D intersects $C^{(i)}$ transversally, yet shares a tangent with $C^{(j)}$. We can overcome this by consider the quotient $\frac{\nu_p(D,C_i)}{\nu_p(C_i)}$ instead of $\nu_p(C^{(i)}, D)$. This *contact exponent* of $C^{(i)}$ and D at p, is a better way to measure the contact between D and different components of C at p.

Definition 3.2.12. Consider a curve C in $\mathbb{C}^2$, with irreducible components C_i. Let p be a point of C.

I. For any smooth curve D through p, we define the contact exponent of D and C at p as

$$\delta_p(D, C) = \min_i \frac{\nu_p(D, C_i)}{\nu_p(C_i)}$$

II. The first contact exponent of C at p is

$$\delta_p(C) = \sup_D \delta_P(D, C_i)$$

III. A smooth curve D through p has maximal contact with C at p if

$$\delta_p(D, C) = \delta_p(C)$$

Next, we investigate the relationship between the contact exponent of C and $D = \zeta(y)$ at p and the Newton polygon of f, $\mathfrak{N}(f)$. From Lemma 2.4.21, we know that if $f = f_i \dots f_r$ is the decomposition of f into irreducible factors, then $\mathfrak{N}(f)$ consists of the $\mathfrak{N}(f_i)$ suitable joined together, and $\mathfrak{N}(f_i)$ consists of a single segment. We use the slopes of these segments as a way of measure how much a σ-process improves a singularity. Suppose that the slope of this segment is $1/\gamma_i$. Then the slope of the steepest segment of $\mathfrak{N}(f)$ is $\frac{1}{\gamma}$ with

$$\gamma = \min_i \gamma_i.$$

Lemma 3.2.13. Let

$$C = \zeta\left(f(x, y) = \sum a_{\alpha\beta} x^\alpha y^\beta\right)$$

and $f = f_i \dots f_r$ be the decomposition of f into irreducible factors. Let the slope of the steepest segment of $\mathfrak{N}(f)$ be $\frac{1}{\gamma}$, that is

$$\gamma = \min_i \gamma_i$$

where γ_i is the slope of the Newton polygon of f_i.

Then

$$\gamma = \delta_{(0,0)}(D, C)$$

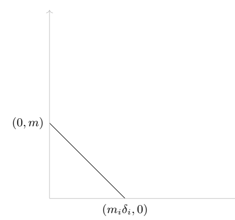
Proof. Recall Definition 3.2.12

$$\delta_{(0,0)}(D, C) = \min_i \delta_{(0,0)}(D, C_i)$$

so we need only show that

$$\gamma_i = \delta_{(0,0)}(D, C_i).$$

By our assumptions, the y-axis is not tangential to C_i, and f_i is y-general of order $m_i = \nu_{(0,0)}(C_i)$ for all i. By Lemma, 2.4.21 $\mathfrak{N}(f_i)$ is just a single segment of the following form:



Next, we compute the intersection multiplicity of D and C_i at $(0, 0)$, which is

$$\dim_{\mathbb{C}} \mathbb{C}[x.y] \Big/ \langle y, f_i \rangle .$$

Note that $f_i(x, y) - f_i(x, 0)$ is a power series divisible by y, and thus is contained in the ideal $\langle y, f_i \rangle$, and $f_i(x, 0) = x^{m_i\gamma_i} h(x)$ with $h(0) \neq 0$. So h is a unit in $\mathbb{C}\{x, y\}$ and

$$\langle y, f_i \rangle = \langle y, f_i(x, 0) \rangle = \langle y, x^{m_i\gamma_i} \rangle.$$

Hence,

$$\nu_{(0,0)}(D, C_i) = \dim_{\mathbb{C}} \mathbb{C}[x.y] \Big/ \langle y, f_i \rangle = m_i\gamma_i$$

and

$$\delta_{(0,0)}(D, C_i) = \frac{\nu_{(0,0)}(D, C_i)}{\nu_{(0,0)}(C_i)} = \frac{m_i\gamma_i}{m_i} = \gamma_i.$$

$\square$

Using this Lemma, we now derive conditions under which D has maximal contact with C at p, by determining when D does *not* have maximal contact with C at p.

Proposition 3.2.14. The curve $D = \zeta(y)$ has non-maximal contact with the curve C at $(0,0)$ if and only if the homogeneous part $F(x,y)$ of $f(x,y)$ is of the form

$$F(x,y) = c(y - \lambda x^{\gamma})^m.$$

Proof. We begin by assuming that D does not have maximal contact with C at $(0,0)$, so there is a smooth curve D' such that

$$\delta_{(0,0)}(D', C) > \delta_{(0,0)}(D, C). \tag{31}$$

From our assumptions, the curve D' is not tangential to the x-axis at $(0,0)$, as otherwise $\delta_0(D', C) = 1$ since the x-axis is not tangential to C.

Thus, by the Implicit Function Theorem, we can describe D' by an equation

$$g(x,y) = y - \sum_{i=1}^{\infty} b_i x^i,$$

and by considering $\mathfrak{N}(g)$, we know that $b_i = 0$ for all $i < d := \delta_{(0,0)}(D, D')$, and $b_d \neq 0$.

Next we show that

$$d = \delta_{(0,0)}(D, D') = \delta_{(0,0)}(D, C),$$

by showing that

$$\delta_{(0,0)}(D, D') < \delta_{(0,0)}(D, C) < d\delta_{(0,0)}(D, D').$$

First assume

$$d = \delta_{(0,0)}(D, D') < \delta_{(0,0)}(D, C) \tag{32}$$

and then perform the coordinate transform

$$x' = x$$

$$y' = y - \sum_i b_i x^i \tag{†}$$

and put

$$g'(x', y') = g\left(x', y' + \sum_i b_i x'^i\right).$$

In the new coordinate system, D' consists of the points (x', y') with $y' = 0$ and $\mathfrak{N}(g')$ is the Newton polygon associated to D' after the coordinate transformation.

We now discuss the relation between $\mathfrak{N}(g)$ and $\mathfrak{N}(g')$. The monomials $x^\alpha y^\beta$ of g become monomials of the form $x'^\alpha(y' + \sum b_i x'^i)$ in the expression of g'. We know that $b_i = 0$ for all $i < \delta_{(0,0)}(D, D')$, and thus the points in the carrier of g' which are associated with the term $a_{\alpha\beta}x^\alpha y^\beta$ all lie on, above, or to the right of the line through (α, β) with slope $\frac{-1}{d}$.

Thus the term $(x')^{m\delta_{(0,0)}(D,D')}(y')^0$ is only related to the term y^m in the expression of g, with coefficient $b_d^m \neq 0$.

Hence, $\mathfrak{N}(g')$ is the segment between $(0, m)$ and $(md, 0)$ which has slope $\frac{-1}{d}$. By Lemma 3.2.13 $\delta_{(0,0)}(D', C) = \frac{-1}{d}$.

But, (31) and (32) yield

$$\delta_{(0,0)}(D', C) = d < \delta_{(0,0)}(D, C) < \delta_{(0,0)}(D', C)$$

a contradiction, as D' has greater contact with C than D.

Next, we assume that

$$d = \delta_{(0,0)}(D, D') > \delta_{(0,0)}(D, C) \tag{33}$$

Again performing the transformation (†), the points of $\Delta(g')$ stemming from the monomial $a_{\alpha\beta}x^\alpha y^\beta$ of g all lie on or above the line through (α, β) with slope $\frac{-1}{d}$. These lines are not as steep as the first segment of $\mathfrak{N}(g)$, and thus the points of $\Delta(g)$ on this segment are also in $\Delta(g')$, so that the slope of the first segment of $\mathfrak{N}(g')$ has slope $\frac{-1}{\gamma}$.

By Lemma 3.2.13

$$\delta_{(0,0)}(D', C) = \delta_{(0,0)}(D, C)$$

in contradiction with (31).

Thus

$$\delta_{(0,0)}(D, D') = \delta_{(0,0)}(D, C) = \gamma.$$

The action of the coordinate transform (†) on the Newton polygon is as follows: the term $a_{\alpha\beta}x^\alpha y^\beta$ generates points in $\Delta(g')$ which are on or above the line with gradient $\frac{-1}{\gamma}$ through the point $(\alpha.\beta)$. From (31), the steepest segment of $\mathfrak{N}(g')$ is flatter than the first segment of $\mathfrak{N}(g)$, and so (†) kills all the points of $\Delta(g)$ except $(0, m)$ which lie on the first segment of $\mathfrak{N}(g)$.

Hence $F(x,y) = cy'^m +$ higher order terms, and by (†),

$$F(x,y) = c(y - b_\delta x^\gamma)^m$$

Conversely, we show that if

$$F(x,y) = c(y - \lambda x^\gamma)^m$$

then D has non-maximal contact with C.

Take the curve $D' = \zeta(y - \lambda x^\delta)$, which is smooth and passes through $(0,0)$. After the coordinate transformation

$$x' = x$$
$$y' = y - \lambda x^\delta$$

D' is given by $y' = 0$ and the term $a_{\alpha\beta}x^\alpha y^\beta$ of general $f(x,y)$ is transformed to $a_{\alpha,\beta}x'^\alpha(y'\lambda x'^\delta)^\beta$.

Thus

$$f' = a_{0m}y'^m + \sum_{\alpha+\delta\beta>\gamma m} a_{\alpha\beta}x'^\alpha(y' - \lambda x'^\gamma)^\beta.$$

The points in $\Delta(h = \sum_{\alpha+\delta\beta>\gamma m} a_{\alpha\beta}x'^\alpha(y' - \lambda x'^\gamma)^\beta)$ corresponding to the $a_{\alpha\beta}(x')^\alpha(y' + \lambda(x')^\gamma)^\beta$ such that $\alpha + \gamma\beta > \gamma m$ are all above the line through $(0,m)$ with gradient $\frac{-1}{\gamma}$, and thus this line meets $\mathfrak{N}(f')$ only at $(0,m)$. Lemma 3.2.13 implies $\delta_{(0,0)}(D', C) > \gamma = \delta_{(0,0)}(D, C)$, which means D' has greater contact with C than D, and so D does not have maximal contact with C at p. □

3.2(b). Contact Exponent at a Singularity. Now that we understand the conditions under which the x-axis has maximal contact with a singular curve C at the singularity $(0,0)$, we show that the contact exponent of C at a singularity is finite. Then we show that given a smooth curve D through $(0,0)$ which has maximal contact with C, their pre-images have maximal contact for a finite number σ-processes. To show this, we introduce *infinitely near* points of a singularity, and use these to obtain a bound on the number of σ-processes required before the multiplicity of the singularity decreases.

We first show that the contact exponent of C at a regular point p is infinite. This is equivalent to no curve having maximal contact with C at p.

Lemma 3.2.15. If the point p is a singularity of the curve C, then the first contact exponent $\delta_p(C)$ of C at p is finite.

Proof. We show that if $\delta_p(C)$ is infinite, then p is a regular point of C.

Without loss of generality, assume that $p = (0,0)$, $\delta_p(C) = \infty$, and that $D^{(0)}$ is a smooth curve through p. We choose coordinates such that $D^{(0)} = \zeta(y)$ and the polynomial $f(x,y)$ of which C is the zero set is a Weierstrass polynomial of degree m, that is the coefficient of $x^0 y^m$ is 1, and we consider f as a polynomial in y with coefficients that are polynomials in x.

$$f(x,y) = y^m + a_{m-1}(x)y^{m-1} + \ldots + a_m(x)$$

Thus, every point of $\Delta(f) \setminus \{(0,m)\}$ lies below the line $\beta = m-1$.

Now, if $D^{(0)} = C$, then C is non-singular, so we assume $D^{(0)} \neq C$. Then $D^{(0)}$ has non-maximal contact with C at $(0,0)$.

By Lemma 3.2.14

$$f(x,y) = a_{0m}(y - b_0 x^{\gamma_0})^m + \sum_{\alpha + m\gamma_r \beta > \gamma_r m} a_{\alpha\beta} x^\alpha y^\beta \quad \gamma \in \mathbb{N}$$

and the curve $D^{(1)} = \zeta(y - b_0 x^{\gamma_0})$ has better contact with C at p.

Thus, we perform the coordinate transformation $x' = x$, $y' = y - b_o x^\gamma$, so that $D^{(1)}$ is given by $y' = 0$.

If $D^{(1)} \neq C$, we may apply this procedure to $D^{(1)}$ and obtain a curve $D^{(2)}$ which has better contact with C at p, and another constant γ_2.

Iterating, we obtain a sequence (γ_i) which is monotonically increasing, and

$$\sum_{\alpha + m\gamma_r \beta > \gamma_r m} a_{\alpha\beta} x^\alpha y^\beta$$

becomes divisible by an arbitrarily large power of x, when r is sufficiently large. This follows from the observation that if (γ_i) is a finite sequence, then $D_k = C$ for some $k \in \mathbb{N}$. Otherwise, we have for each r the expression

$$f(x,y) = a_{0m}(y_r - b_r x^{\gamma_r})^m + \sum_{\alpha + \gamma_r \beta > m\gamma_r} a^r_{\alpha\beta} x^\alpha y_r^\beta.$$

By expanding $(y_r - b_r x^{\gamma_r})^m$, we see that the points in

$$\Delta\left(\sum_{\alpha+\gamma_r\beta>\gamma_r m} a^{(r)}_{\alpha\beta}x^\alpha y_r^\beta\right)$$

are all on or above the line through $(0, m)$ which has gradient $\frac{-1}{\gamma_r}$ and below the line $\beta = m - 1$. Thus, the sequence of remainders converges to 0 and

$$f(x, y) = \sum a_{0m}\left(y - \sum b_i x^{\delta_i}\right)^m \tag{34}$$

which implies that C is regular at $(0, 0)$.

□

With the necessary and sufficient condition for D to have maximal contact with C at p from Proposition 3.2.14, we investigate what happens to D under a σ-process centred at $p = (0, 0) \in \mathbb{C}^2$. We do so by examining the points in the sequence of strict transforms which are in the pre-image of $(0, 0)$: the points of the blow up which are 'infinitely near' to the point $(0, 0)$. The multiplicities of these points in the pre-images C_i of C tell us by how much each σ-process has resolved the singularity. We will show that there are only finitely many infinitely near points with the same multiplicity as the original singularity, and thus a finite sequence of σ-processes reduces the multiplicity. Repeating, we obtain a bound on the number of σ-processes required before we obtain a non-singular curve.

Definition 3.2.16. Take a curve C in $\mathbb{C}^2$, and a sequence of σ-processes

$$\mathbb{C}^2 \xrightarrow{\Upsilon_k} \mathbb{C}^2 \xrightarrow{\Upsilon_{k-1}} \dots \xrightarrow{\Upsilon_2} \mathbb{C}^2 \xrightarrow{\Upsilon_1} \mathbb{C}^2,$$

with Υ_1 centred at $p \in C$. Let $E_i = \Upsilon_i(p)$, and C_i be the strict pre-image of C under ϕ_i. The points $x \in E_i \cap C_i$ are called infinitely near points of $p \in C$, and those with the additional property that $\nu_x(C_i) = \nu_p(C) =: \nu$ are called ν-tuple infinitely near points of p in C_i.

We begin by investigating the contact between the strict pre-images of the singular curve C and a smooth curve D which has maximal contact with C at the singularity $(0, 0)$. Assume that at $(0, 0) \in C = \zeta(f(x, y))$ with $f(x, y) = \sum a_{\alpha\beta}x^\alpha y^\beta$, the curve $D = \zeta(y)$ has maximal contact with C at $(0, 0)$, that is $\delta_{(0,0)}(C) = \delta_{(0,0)}(D, C)$. Let

$$\Upsilon^{-1} : \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid x = 0\} \longrightarrow \mathbb{C}^2 \setminus \{(x, y) \in \mathbb{C}^2 \mid x = 0\}$$

be a σ-process centred at $(0, 0)$, given by $\Upsilon^{-1}((u, v)) = (v, uv)$ in a neighbourhood of $p = (0, 0)$.

Let p_1 be an infinitely near point of $(0,0)$, and the strict pre-images of C and D under Υ^{-1} be C_1 and D_1 respectively. Then, C_1 is given by $f'(u,v) = \sum a_{\alpha\beta} v^{\alpha+\beta-m} u^{\beta}$. The degree of $x^{\alpha+\beta-m} y^{\beta}$ is $\alpha + 2\beta - m$. Thus the multiplicity of p_1 is m if $a_{\alpha\beta} = 0$ for $\alpha + 2\beta < 2m$, which implies that this occurs when the slope of the steepest segment of the Newton Polygon of C is less than $\frac{-1}{2}$.

Thus $\delta_{(0,0)}(D,C) \geq 2$.

Under a σ-process Υ^{-1}, the segment l of $\mathfrak{N}(f)$ between $(0,m)$ and (α,β) corresponds to the segment l' between $(0,m)$ and $(\alpha + m\beta, \beta)$ of $\mathfrak{N}(f')$. The slope of l is $\frac{-1}{\alpha/(m-\beta)}$ and the gradient of l' is $\frac{-1}{\alpha/(m-\beta)-1}$.

Thus, if the slope of the steepest segment of $\mathfrak{N}(f)$ is $\frac{-1}{\gamma}$, then the gradient of the steepest segment of $\mathfrak{N}(f')$ is $\frac{-1}{\gamma-1}$. Applying Lemma 3.2.13, we obtain

$$\delta_{p_1}(D_1, C_1) = \delta_{(0,0)}(D,C) - 1 \tag{35}$$

Next, we consider the contact exponent of D_1 and C_1 at p_1. We know that the quasi-homogeneous component of $f(x,y)$ is found through the steepest segment of $\mathfrak{N}(f_1)$ and is of the form

$$F_1(v,u) = \frac{F(v,u)}{v^m}.$$

If D_1 does not have maximal contact with C_1 at p_1, we know that $F_1(v,u) = (u - \lambda v^{\delta})^m$ is binomial by Lemma 3.2.14. Then

$$F(x,y) = x^m (y/x - \lambda x^{\delta})^m = (y - \lambda x^{\delta+1})^m$$

which by the same Lemma implies that D has non-maximal contact with C at p, a contradiction. Thus we obtain the following proposition:

Proposition 3.2.17. If D has maximal contact with C at p and $\delta_p(D,C) \geq 2$, then there is a $p_1 \in C_1$ such that $\sigma(p_1) = p$ and $\nu_{p_1}(C_1) = \nu_p(C) = m$. Moreover, given $p_i \in D_1$, D_1 has maximal contact with C_1 at p_1, and

$$\delta_{p_1}(C_1) = \delta_{p_1}(D_1, C_1) = \delta_{(0,0)}(D,C) - 1 \tag{36}$$

Using Lemma 3.2.15, we can prove that the number of infinitely near points of p with the same multiplicity is bounded.

Proposition 3.2.18. If C is a curve in $\mathbb{C}^2$, and $p \in C$ is such that $\nu_p(C) = m$, then the number of m-tuple infinitely near points of p is the integer part $\lfloor \delta_p(C) \rfloor$ of the first characteristic exponent of C at p.

Proof. Without loss of generality, we assume that $p = (0, 0)$.

If $m = 1$, then $\delta_{(0,0)}(C) = \infty$, and successive σ-processes generate regular curves, and so this is trivially true.

Thus, we assume that $m > 1$. By Lemma 3.2.15, there is a smooth curve D which has maximal contact with C at p. From Proposition 3.2.17, we know that after $\lfloor \delta_{(0,0)}(C) \rfloor$ σ-processes we arrive at a curve C' and point p' such that $\delta_{p'}(C') \leq 2$. Any singularities which are generated by this sequence of σ-processes all have multiplicity no greater than m. Hence, the number of m-tuple points is $\lfloor \delta_p(C) \rfloor$

□

This shows that any singularities generated by a finite sequence of σ-processes each have lower multiplicity than the original singular point $p \in C$. By induction on $\nu_p(C)$, we arrive at a curve which is non-singular after a finite sequence of σ-processes. As a singular curve C only has finitely many singularities and σ-processes do not affect other singularities, we can perform a sequence of blow ups at each singular point $p_i \in C$, each of which resolves the singularity at p_i.

Summarising we have:

Theorem 3.2.19. The singularities of an irreducible curve $C \subset \mathbb{CP}^2$ can be resolved by a finite sequence of σ-processes.

3.3. Algorithmic Approaches

The use of σ-processes to resolve the singular points of a curve requires long series of calculations, which include choices. While for points of low multiplicity this is of no great concern, as the multiplicity of the singularity increases, we must address these issues. Thus, we seek an algorithm which is equivalent to a sequence of σ-processes in resolving a singularity of a curve $C \subset \mathbb{C}^2$. We use the algorithm developed in [2], and as blow ups and σ-process are local, it is sufficient to provide it for curves in $\mathbb{C}^2$.

Given a curve $C = \zeta(f(x, y)) \in \mathbb{C}[x, y]$, we partition $f(x, y)$ into homogenous components as follows:

$$f(x, y) = f_r(x, y) + f_{r+1}(x, y) + \ldots + f_n(x, y),$$

where $f_k \in \mathbb{C}[x, y]$ is a homogeneous polynomial of degree k.

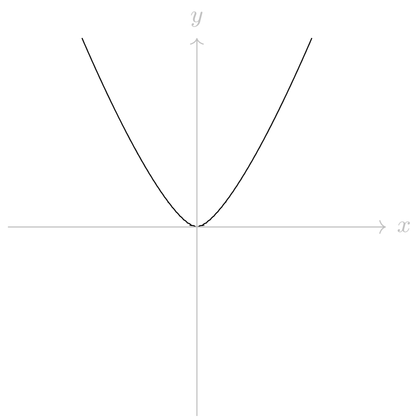
Remark 3.3.20. The multiplicity of the point $(0,0) \in C$ is r, and n is the degree of $f(x,y)$.

Defining

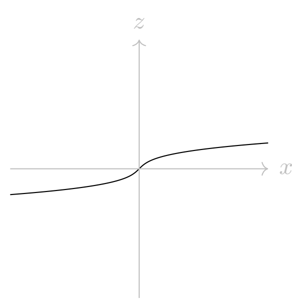
$$f'(x,z) = f_r(1,z) + x^1 f_{r+1}(1,z) + \ldots + x^{n-r} f_n(1,z) \tag{37}$$

the curve $C' = \zeta(f'(x,z))$, is a plane curve in $\mathbb{C}^2 \setminus \{\Upsilon^{-1}(P)\}$, which is birationally equivalent to C in $\mathbb{C}^2$. If we are resolving an ordinary singularity at $(0,0)$, that is the number of tangents to C is equal to $\nu_{(0,0)}(C)$, then this algorithm generates a non-singular plane curve.

Example 14. Take the curve $C = \zeta(x^4 - x^2y - y^3)$,



Applying the algorithm, we note that $f_3 = -x^2y - y^3$ and $f_4 = x^4$, so we have $f'(x,z) = -1z - z^3 - x \times 1 = x - z - z^3$, which is a non-singular curve.



We also want to resolve singularities which are non-ordinary multiple points of the curve $C \subset \mathbb{CP}^2$ with an algorithm, as in Example 14. The best outcome would be to use the same algorithm. The following example shows that this algorithm is not sufficient.

Example 15. Take $C = \zeta(y^2 - x^5)$,

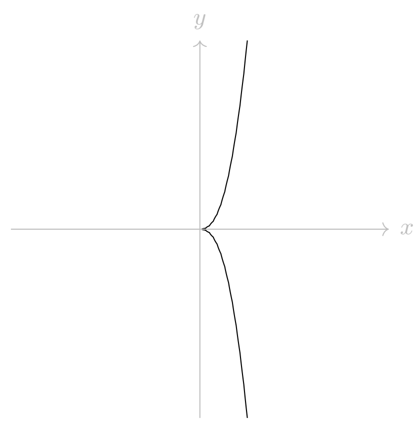


FIGURE 17. $C = \zeta(y^2 - x^5)$

Then, the algorithm yields $C' = \zeta(z^2 - x^3)$, which is still singular.

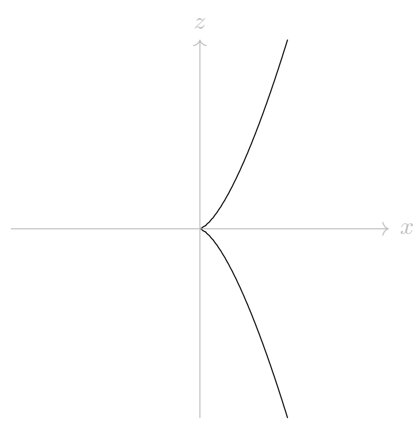


FIGURE 18. $C' = \zeta(z^2 - x^3)$

From this example, we see that the algorithm developed does not necessarily give us a standard resolution if the original curve has a non-ordinary singularity. To overcome this issue, we introduce quadratic transformations of $\mathbb{CP}^2$, which transform non-ordinary multiple points into ordinary multiple points.

With coordinates $[X : Y : Z]$ on $\mathbb{CP}^2$, we call the points $P^1 = [0 : 0 : 1]$, $P^2 = [0 : 1 : 0]$, $P^3 = [1 : 0 : 0]$ fundamental points, and define the lines $L^1 = \zeta(Z)$, $L^2 = \zeta(Y)$ and $L^3 = \zeta(X)$, called fundamental lines. Let $U = \mathbb{CP}^2 \setminus \{P^1, P^2, P^3\}$. We define the standard quadratic transformation $\mathfrak{Q}$, as we can obtain other quadratic transformations by composing $\mathfrak{Q}$ with a change of coordinates T [1].

Definition 3.3.21. The standard quadratic transformation of $\mathbb{CP}^2$ is the map

$$\mathfrak{Q} : U \longrightarrow \mathbb{CP}^2, \quad [X : Y : Z] \longmapsto [YZ : XZ : XY] \tag{38}$$

Lemma 3.3.22. Given a curve $C \subset \mathbb{CP}^2$, $\mathfrak{Q}(C)$ is birationally equivalent to C.

Proof. It is enough show that there is an open subset U of $\mathbb{CP}^2$ such that $\mathfrak{Q} : U \longrightarrow U$ is an isomorphism.

Noting that

$$\mathfrak{Q}^2[X : Y : Z] = \mathfrak{Q}[YZ : XZ : XY] = [XYZX : XYZY : XYZZ] = [X : Y : Z] \tag{39}$$

we see $\mathfrak{Q}^{-1} = \mathfrak{Q}$, and thus

$$\mathfrak{Q} : U \xrightarrow{\cong} U$$

Hence $\mathfrak{Q}$ is a birational map between C and $\mathfrak{Q}(C)$.

□

If $C = \zeta(F)$, where $F[X : Y : Z] \in \mathbb{C}[X : Y : Z]$ is a homogenous polynomial of degree n, then what is the homogenous polynomial $F'[X : Y : Z]$ associated with C'?

We start by considering the action of $\mathfrak{Q}$ on F. Given $C \subset \mathbb{CP}^2$ call

$$F^{\mathfrak{Q}} = F[YZ : XZ : XY]$$

the *algebraic transform* of C [2].

Remark 3.3.23. $F^{\mathfrak{Q}}$ has degree $2n$.

Next, we decompose F as follows: let $r^1 = m_{P^1}(C)$, $r^2 = m_{P^2}(C)$ and $r^3 = m_{P^3}(C)$, then, we consider F to be the sum of homogeneous polynomials of degree i multiplied by a factor of Z^{n-i}.

$$F[X : Y : Z] = \sum_{i=r^1}^{n} F_i[X : Y]Z^{n-i} \tag{310}$$

Then, $F^{\mathfrak{Q}}$ is given by

$$\sum_{i=r^1}^{n} F_i[ZY : XZ](XY)^{n-i} \tag{311}$$

Clearly, we can factor out Z^{r^1}, Y^{r^2} and X^{r^3}, and so

$$F^{\mathfrak{Q}} = Z^{r^1}Y^{r^2}X^{r^3}F'[X : Y : Z]. \tag{312}$$

We call $F'[X : Y : Z]$ in this expression the *proper transform* of F.

For $\mathfrak{Q}$ to be useful in resolving singularities of curves, applying $\mathfrak{Q}$ must reduce a non-ordinary singularity of a $C \subset \mathbb{CP}^2$ to an ordinary curve without introducing new non-ordinary singularities. For $\mathfrak{Q}$ to not introduce non-ordinary singularities, C must be such that none of the fundamental lines are tangential to C at a fundamental point, and L^1 intersects C transversally in n distinct non-fundamental points, whilst L^2 and L^3 intersect C transversally in $n - r^1$ distinct non-fundamental points each.

Whilst these conditions appear to restrict the utility of quadratic transformations, we will show that through a change of coordinates, any curve in $\mathbb{CP}^2$ can be transformed to satisfy the requirements.

Definition 3.3.24. We say $C \subset \mathbb{CP}^2$ is in *excellent* position if it satisfies the following:

(a) none of the fundamental lines are tangential to C at a fundamental point,

(b) L^1 intersects C transversally in n distinct non-fundamental points

(c) L^2 and L^3 intersect C transversally in $n - r^1$ distinct non-fundamental points each.

If C is in excellent position, then with $C' = \zeta(F')$ as defined above, we obtain the following Lemma specifying what the singular points of C' are:

Lemma 3.3.25. If C is in excellent position, then C' has the following singularities:

(a) those points in $\mathfrak{Q}(C) \cap U$ which correspond to multiple points, with the multiplicity preserved, as well as the type of multiple point

(b) P^1, P^2, P^3 are ordinary multiple points of C', with multiplicities n, $n - r^1$ and $n - r^1$ respectively

(c) Any non-fundamental points on $C' \cap L^1$, say $p_1, \dots, p_s$, with $\nu_{p_i}(C') \le \nu_{p_i}(C', L^1)$ and

$$\sum \nu_{p_i}(C', L^1) = r \tag{313}$$

Proof. For (a), note that $C' \cap U$ and $C \cap U$ are isomorphic, and so from Theorem 2.3.18, the multiplicities of corresponding points are the same. No fundamental line is tangential to C at a fundamental point, and thus none are tangent fo C' at a fundamental point either.

Furthermore,

$$\sum \nu_{p_i}(C' \cap L) = \sum \nu_{p_i}(\zeta(F_r(Y, X) \cap L) = r$$

with F_r the degree r homogenous component of F. This yields (c) and (b) follows by applying this to $\mathfrak{Q}(C)$ and $\mathfrak{Q}(\mathfrak{Q}(C)) = C$. □

Now that we know which points of the strict transform C' of C are singular, we note that given any curve $C \subset \mathbb{C}^2$ there is a change of coordinates T such that the curve $T(C)$ is in excellent position and $T([0:0:1]) = P$, with P the point of interest in C.

Summarising we obtain

Theorem 3.3.26. Any irreducible curve $C \subset \mathbb{CP}^2$ may be transformed into a birationally equivalent curve with singularities which are only ordinary multiple points by a finite sequence of quadratic Transformations.

And thus we can use the algorithm and quadratic transformations to obtain a standard resolution of a singularity of $C \subset \mathbb{CP}^2$.

3.4. Resolving several singularities

As σ-processes are local, they do not affect other singularities, and so we may simultaneously blow up several singular points p_i of the curve C. To resolve several singular points $p_1, \ldots p_t$ of a curve $C \subset \mathbb{C}^2$, we choose coordinates on $\mathbb{CP}^2$ such that all the p_i lie in the first coordinate chart, and consider these points in $\mathbb{C}^2$.

We define a map for each $p_i = (a_i, b_i)$, an analogue of the construction in Definition 3.1.3:

$$f_i : \mathbb{C}^2 \setminus \{p_i\} \longrightarrow \mathbb{CP}^1, \quad (x, y) \longmapsto [x - a_i : y - b_i]$$

and use these to construct

$$\mathcal{F} = (f_i, \ldots, f_t) : \mathbb{C}^2 \setminus \{p_i\} \longrightarrow \mathbb{CP}^1_1 \times \ldots \times \mathbb{CP}^1_t.$$

Let $\mathcal{G} \subset \mathbb{C}^2 \times \mathbb{CP}^1_1 \times \ldots \times \mathbb{CP}^1_t$ be the graph of f.

Lemma 3.4.27. The closure of $\mathcal{G}$ is the set

$$\mathcal{B} := \zeta\left(\{U_i(y-b_i) - V_i(x-a_i) \mid i = 1, \ldots, t\}\right) \subset \mathbb{C}^2 \times \mathbb{CP}^1_1 \times \ldots \times \mathbb{CP}^1_t$$

Proof. The only points in $\mathcal{B} \backslash \mathcal{G}$ are those which correspond to the singular points $p_i \in C \subset \mathbb{CP}^2$, which are the limit points of $\mathcal{G}$ not in $\mathcal{G}$. Thus, $\overline{\mathcal{G}} = \mathcal{B}$. □

Let ρ be the restriction to $\mathcal{B}$ of the projection from $\mathbb{CP}^2 \times \mathbb{CP}^1_1 \times \ldots \times \mathbb{CP}^1_t$ to $\mathbb{CP}^2$.

Now take a curve $C \subset \mathbb{CP}^2$ with singularity $(0,0)$, and call $\rho^{-1}(C)$ the strict pre-image of C, and $E = \rho^{-1}((0,0))$ the exceptional line.

To ensure that we obtain another plane curve, and can repeat the process if necessary, we make a definition analogous to Definition 3.1.6:

Definition 3.4.28. Let $\widetilde{\pi}$ be the projection from $\mathbb{C}^2 \times \mathbb{CP}^1 \times \ldots \mathbb{CP}^1$ to $\mathbb{C}^2$ given by

$$\widetilde{\pi}((x, xv), [u_1 : v], \ldots, [u_t : v]) = (x, v).$$

Then, define

$$\Upsilon = \mathcal{F}^{-1} \circ \widetilde{\pi}^{-1} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2, \tag{314}$$

and call $\Upsilon^{-1}(C)$ the strict transform of C.

The definition of a standard resolution as in 3.1.10 is still the appropriate definition, and by Theorem 3.2.19, we know that each singularity can be resolve in a finite sequence of σ-processes, and so we obtain the corollary

Corollary 3.4.29. For any irreducible curve $C \subset \mathbb{CP}^2$ with singularities p_i, we can resolve the singularities through a finite sequence of σ-processes, and the upper bound on the number required is related to the maximum of the multiplicities $\nu_{p_i}(C)$.

3.5. Reducible Curves

The assumption of irreducibility has been persistent throughout the preceding discussion. In this section, we will show that this assumption can be relaxed, allowing us to further strengthen Theorem 3.3.26.

To resolve the singularity of a reducible curve $C = \cup_i C_i$, we blow up each irreducible component C_i, and thus the only notion which requires modification is that of a standard resolution.

Definition 3.5.30 (Standard resolution for reducible curves)**.** Take a reducible curve $C = \cap_{i=1}^{r} C_i$ in $\mathbb{C}^2$, $r \geq 2$. Let $\mathbb{C}^2_j \xrightarrow{\Upsilon_j} \mathbb{C}^2_{j-1} \xrightarrow{\Upsilon_{j-1}} \ldots \xrightarrow{\Upsilon_2} \mathbb{C}^2_1 \xrightarrow{\Upsilon_1} \mathbb{C}^2$ a sequence of σ-processes, with $E^{(i)} = (\Upsilon_1 \circ \ldots \circ \Upsilon_i)^{-1}(0,0)$ the i^{th} exceptional line, and $C^{(i)} = \overline{(\Upsilon_1 \circ \ldots \circ \Upsilon_i)^{-1}(C \setminus \{(0,0)\}}$ the i^{th} strict transform of C.

Take $\mathbb{C}^2_{i+1} \xrightarrow{\Upsilon_{i+1}} \mathbb{C}^2 i$ as the blow up all points $p_i \in C^{(i)} \cap E^{(i)} \subset \mathbb{C}^2 i$ such that $\nu_{p_i}(C^{(i)}) > 1$ or $\nu_{p_1}(C^{(i)}, E^{(i0)}) > 1$ or $C^{(i)}$ intersects two components of $E^{(i)}$ simultaneously.

Then a standard resolution is $\mathbb{C}^2_k \xrightarrow{\Upsilon_k} \mathbb{C}^2_{k-1} \xrightarrow{\Upsilon_{k-1}} \ldots \xrightarrow{\Upsilon_2} \mathbb{C}^2_1 \xrightarrow{\Upsilon_1} \mathbb{C}^2$ such that all the branches of $C^{(k)}$ are smooth, $C^{(i)} \cap C^{(j)} = \emptyset$ for all i, j, and the intersections of $C^{(k)}$ and $E^{(k)}$ do not occur where multiple branches of $E^{(k)}$ meet and are transversal.

Example 16. Take the reducible curve $C = \zeta((Y^2Z - X^3)(Y^2Z - X^3 - X^2Z)) \subset \mathbb{CP}^2$, with irreducible components

$$C_1 = \zeta(Y^2Z - X^3)$$
$$C_2 = \zeta(Y^2Z - X^3 - X^2Z)$$

C has a singularity at $[0:0:1]$.

We choose an affine chart by letting $Z = 1$, and perform the first σ-process.

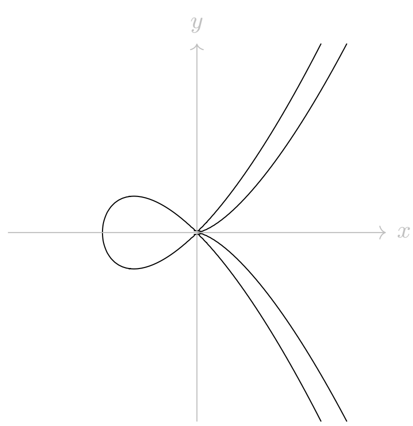


FIGURE 19. $\zeta((y^2 - x^3)(y^2 - x^3 - x^2))$

A first σ-process by substituting xy for y yields the strict transform $(y^2 - x)(y^2 - x - 1)$ and exceptional line $E_1 = \zeta(x)$

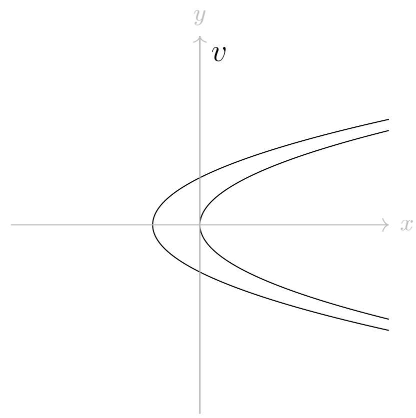


FIGURE 20. $\zeta((v^2-x)(v^2-x-1))$

We pass to $\mathbb{CP}^2$ again, and obtain the curve $\zeta((Y^2-Z)(Y^2-XZ-Z^2))$. Choosing the affine chart $X=1$, we obtain $(y^2-z)(y^2-z-z^2)$, and perform a σ-process by replacing y with yz, which yields the strict transform $\zeta((yz-1)(yz-z-1))$, and exceptional line $E_1=\zeta(x)$ and $E_2=\zeta(z)$. Here E_1 is the line at infinity.

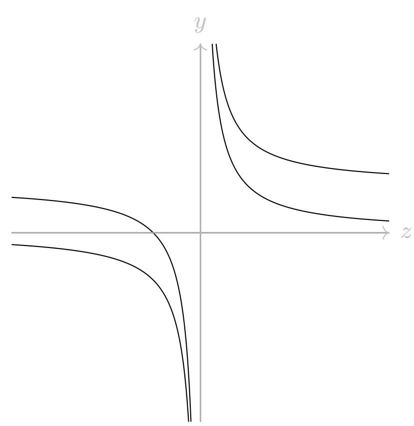


FIGURE 21. $\zeta((yz-1)(yz-z-1))$

Passing to $\mathbb{CP}^2$ again, we obtain the curve $\zeta((YZ-X^2)(YZ-XZ-X^2))$. Choosing the affine chart $Z=1$ yields$(y-x^2)(y-x-x^2)$, and the exceptional lines $E_1=\zeta(x)$ and $E_2=\zeta(z)$, where E_2 is the line at infinity.

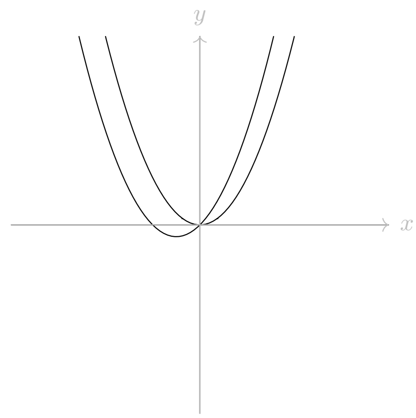


FIGURE 22. $\zeta((y - x^2)(y - x - x^2))$

We perform another σ-process, substitution xy for y, and obtain the strict transform as the zero set of $(y - x)(y - x - 1)$ and exceptional lines $E_1 = \zeta(x)$ and $E_2 = \zeta(z)$, where E_2 is the line at infinity.

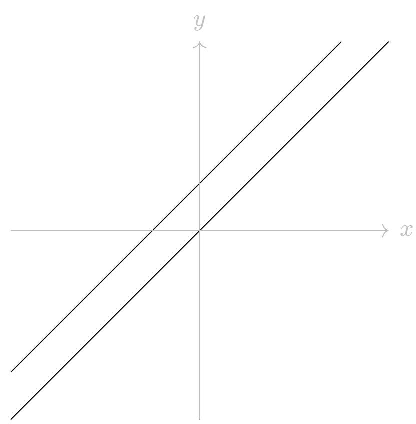


FIGURE 23. $\zeta((v - x)(v - x - 1))$

This is a standard resolution.

Having extended blow ups and σ-processes to reducible curves allows us to strengthen Theorem 3.2.19:

Theorem 3.5.31. Given any curve $C \subset \mathbb{CP}^2$, we can resolve the singularities through a finite sequence of σ-processes.

CHAPTER 4

Representing Standard Resolutions

Now is a time for, dare I say it, kindness. I thought being extremely smart would take care of it. But I see I have been found out.

Margaret Edson
W;t

There are several ways to represent information about singularities of a curve $C \subset \mathbb{CP}^2$. Information about the resolution of a singularity is nicely presented by the multiplicity sequence and the resolution graph. Topological information about the singularity is provided by the Puiseux Pairs (including the braid of the singularity, which allows us to classify singularities). The multiplicity sequence represents the multiplicities of the infinitely near points of the singularity, while the resolution graph depicts how the exceptional lines and strict transform generated by a sequence of σ-processes intersect. We first define the multiplicity sequence for irreducible and reducible curves and provide examples. Then, we define the resolution graph for irreducible and reducible curves, providing examples. Finally, we show that the information contained in the multiplicity sequence, resolution graph, and Puiseux characteristic exponents is equivalent.

4.1. Multiplicity Sequences

First we define the multiplicity sequence for a standard resolution of a singular point p on an irreducible curve C [1].

Definition 4.1.1. Let $C \subset \mathbb{CP}^2$ be an irreducible curve, and p a singular point of C. Take

$$\nu_0 = \nu_p(C),$$

and let ν_i be the multiplicity of the associated point p_i in the i^{th} strict transform $C^{(i)}$ of C. Assume that $C^{(n)}$ is the standard resolution of $p \in C$, then $\nu_i = 1$ for all $i \geq n$. The sequence

$$(\nu_0, \nu_1, \ldots, \nu_{n-1}) \tag{41}$$

is the *multiplicity sequence* of the standard resolution $C^{(n)}$.

Example 17. We now consider the multiplicity sequences for the examples in the previous chapter.

In Example 11 with $C = \zeta(y^2 - x^3)$, the standard resolution has multiplicity sequence is $(2, 1, 1)$.

In Example 12 with $C = \zeta(y^2 - x^3 - x^2)$, the standard resolution multiplicity sequence is (2).

In Example 13 with $C = \zeta(y^2 - x^5)$, the standard resolution multiplicity sequence is $(2, 2, 1, 1)$.

We now extend this to reducible curves.

Definition 4.1.2. Let $C = \cap_{j=1}^{r} C_j \subset \mathbb{CP}^2$ be a reducible curve with irreducible components C_j, and p a singular point of C. Take $\nu_0^j = \nu_p^j(C_j)$, and let ν_i be the multiplicity of the associated point p_i in the i^{th} strict transform $C_j^{(i)}$ of C. Assume that $C^{(n)}$ is the standard resolution of $p \in C$, then $\nu_i = 1$ for all $i \geq n$. For each C_j, we have the multiplicity sequence

$$(\nu_0^j, \nu_1^j, \ldots, \nu_{n-1}^j) \tag{42}$$

Then, we can form the system of sequence

$$\begin{array}{l}(\nu_0^1, \nu_1^1, \ldots, \nu_{n-1}^1)\\(\nu_0^2, \nu_1^2, \ldots, \nu_{n-1}^2)\\\cdots\\(\nu_0^r, \nu_1^r, \ldots, \nu_{n-1}^r)\end{array}$$

which we call the *system of multiplicity sequences* of $p \in C$. For each j, there is an $L_j \in \mathbb{N}$ such that the L_j strict transform of $C^{(j)}$ is non-singular, intersects the exceptional lines as required, and does not intersect the strict transforms of the other irreducible components. We call this L_j the *proper length* of the j^{th} multiplicity sequence.

Example 18. We now consider the multiplicity sequences for the standard resolution of

$$C = \zeta((Y^2Z - X^3)(Y^2Z - X^3 - X^2Z))$$

in Example 16:

$$\begin{array}{c}(2, 1, 1)\\(2, 1, 1)\end{array}$$

4.2. Resolution Graphs

We now define the resolution graph for the standard resolution of an irreducible curve [4].

Definition 4.2.3. Let $C \subset \mathbb{CP}^2$ be an irreducible curve, and p a singular point of C. Assume that $C^{(n)}$ is the standard resolution of the singularity $p \in C$, and recall the definitions of the exceptional lines. The *resolution graph* is the weighted graph with vertices the exceptional lines E_i, $1 \leq i \leq n$ and the curve $C^{(n)}$. Two such vertices are connected if the intersection of the curves they represent is non-empty, that is, we connect E_i and E_j if $E_i \cap E_j \neq \emptyset$, and E_i is connected to $C^{(n)}$ if they intersect. For each E_i, the weight is i, and in the graph we use i to represent E_i and $*$ to depict $C^{(n)}$.

Example 19. We now consider the resolution graph for the examples in the previous chapter.

Example 11: the standard resolution of $C = \zeta(y^2 - x^3)$

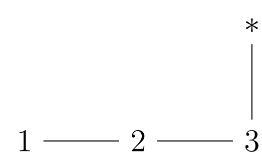


FIGURE 24. Resolution graph of the singularity at $(0,0) \in \zeta(y^2 - x^3)$.

Example 12: the standard resolution of $C = \zeta(y^2 - x^3 - x^2)$

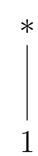


FIGURE 25. Resolution graph of the singularity at $(0,0) \in \zeta(y^2 - x^3 - x^2)$.

Example 13: the standard resolution of $C = \zeta(y^2 - x^5)$

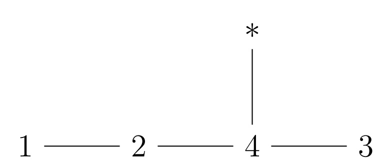


FIGURE 26. Resolution graph of the singularity at $(0,0) \in \zeta(y^2 - x^5)$.

We extend this to reducible curves with the following definition [4].

Definition 4.2.4. Let $C = \cup_{j=1}^{r} C_j \subset \mathbb{CP}^2$ be an irreducible curve, and p a singular point of C. Assume that $C^{(n)}$ is the standard resolution of the singularity $p \in C$, and recall the definitions of the exceptional lines. The *standard resolution* is the weighted graph with vertices the exceptional lines E_i, $1 \leq i \leq n$ and the curve $C^{(n)}$. Two such vertices are connected if the intersection of the curves they represent is non-empty, that is, we connect E_i and E_j if $E_i \cap E_j \neq \emptyset$, and E_i is connected to $C_j^{(n)}$ if they intersect. For each E_i, the weight is i, and in the graph we use i to represent that E_i and $*$ to depict each $C_j^{(n)}$.

A method for constructing the standard resolution of a reducible curve from the graphs of the irreducible components is found in [1]:

Assume we have constructed the standard resolution $\widetilde{G}$ for $\widetilde{C} = \cup_{i=1}^{n-1} C^i$ and let G^n be the standard resolution of the component $C^{(n)}$. The proper lengths $\{L'_j\}_{j=1}^{n-1}$ of the components of $\widetilde{C}$ could increase to L_j when considered as components of C, as with $C^{(n)}$. For each $1 \leq i \leq n$, let $q_i = L_i - L'_i$, and for each $C^{(i)}$ introduce q_i points between the corresponding $*$ and the point it is connected to, with the weights increasing accordingly. Denote the new graphs $\widetilde{G'}$ and $G^{n'}$ respectively. The points of $G^{n'}$ representing exceptional lines have weights given by the number of σ-processes after which they appear.

Assume that Υ_ι is the σ-process which separates $C^{(n)}$ from $\widetilde{C}$. Then the points of $G^{n'}$ with weights less than ι must be identified with points of $\widetilde{G'}$, such that the infinitely near points agree. These identifications can be obtained from the multiplicity sequence.

Now, we construct the standard resolution G for $C = \cup_{i=1}^{n-1} C^i \cup C^{(n)}$ from the modified graphs $\widetilde{G'}$ and $G^{n'}$ as follows:

(1) the points of G are those of $\widetilde{G'}$, and $G^{n'}$ such that $i > \iota$ and the $*$ representing $C^{(n)}$

(2) for points of $G^{n'}$ with index greater than ι, they are connected as in $G^{n'}$, and for index less than or equal to ι they are connected to the corresponding points of $\widetilde{G'}$

(3) points of $\widetilde{G'}$ with weight less than ι remain connected in G if they are connected to the point ι in $\widetilde{G'}$, but the point with same weight from $G^{n'}$ is not connected to ι from $G^{n'}$.

Example 20. We now consider the standard resolution for the standard resolution of

$$C = \zeta((Y^2Z - X^3)(Y^2Z - X^3 - X^2Z))$$

in Example 16.

The proper length for the irreducible component given by $y^2 - x^3 - x^2$ is 1, so we must add 2 points to its standard resolution, and the infinitely near points only coincide at the singularity, so we join the vertices with weights ≤ 1.

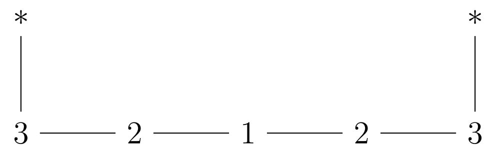


FIGURE 27. Resolution graph of the singularity at $(0,0) \in \zeta((y^2 - x^3)(y^2 - x^3 - x^2))$.

4.3. Relating Multiplicity Sequences and Resolution Graphs

We will show that the multiplicity sequence and resolution graph of an irreducible curve

$$C = \zeta(f(x,y))$$

are equivalent, by exploring their connections with the characteristic Puiseux exponents of $f(x,y)$. We follow [1] closely.

We can represent the Puiseux expansion through the parametrisation

$$\begin{aligned} x &= t^m \\ y &= a_1 t^{k_1} + a_2 t^{k_2} \dots \qquad a_i \neq 0 k_i \in \mathbb{N} \cap \{0\} \end{aligned}$$

and to express $\mu_0(C)$ in terms of m and the k_i. To do so, note that the multiplicity at a point is the intersection multiplicity with a general line at that point. This intersection multiplicity is obtained by substituting the Puiseux expansion into the general line equation. For C, we obtain four cases:

(i) $m \leq k_1 \Rightarrow \nu_{(0,0)}(C) = m$

(ii) $m > k_1 > 0 \Rightarrow \nu_{(0,0)}(C) = k_1$

(iii) $k_1 = 0, m \leq k_2 \Rightarrow \nu_{(0,0)}(C) = m$

(iv) $k_i = 0, m > k_2 \Rightarrow \nu_{(0,0)}(C) = k_2$

We now examine what a blow up does to the Puiseux series in each case:

Case (i): we blow up by substituting $x = v$ and $y = uv$, and have the transformed Puiseux expansion

$$\begin{aligned} u &= a_1 t^{k_1 - m} + a_2 t^{k_2 - m} \dots \\ v &= t^m \end{aligned}$$

Case (ii): we blow up by substituting $x = uv$ and $y = v$, and have the transformed Puiseux expansion

$$\begin{aligned} u &= b_1 t^{m-k_1} + b_2 t^{m-k_2} + \dots \\ v &= a_1 t^{k_1} + a_2 t^{k_2} + \dots \end{aligned}$$

Case (iii): we blow up by substituting $x = v$ and $y - a_i = uv$, and have the transformed Puiseux expansion

$$\begin{aligned} u &= a_2 t^{k_2-m} + a_3 t^{k_3-m} + \dots \\ v &= t^m \end{aligned}$$

Case (iv): we blow up by substituting $x = uv$ and $y - a_1 = v$, and have the transformed Puiseux expansion

$$\begin{aligned} u &= b_1 t^{m-k_2} + b_2 t^{m-k_3} + \dots \\ v &= a_2 t^{k_2} + a_2 t^{k_2} + \dots \end{aligned}$$

For each case, the Puiseux Series for the strict transform again falls into one of the cases, and we can calculate $\nu_0^1(C)$. By repeating this process, we can describe an algorithm for calculating the multiplicity sequence. Through the iterations, we obtain a chain of *'Euclidean algorithms'* of length g, for some g. Hence, we obtain the following Proposition:

Proposition 4.3.5. Take an irreducible curve $C = \zeta(f(x, y))$ with parametric Puiseux expansion

$$\begin{aligned} x &= t^{\Lambda} \\ y &= a_1 t^{\Lambda_1} + a_2 t^{\Lambda_2} + \dots + a_g t^{\Lambda_g} \end{aligned}$$

where we omit terms with non-characteristic exponents. Consider the chain of g algorithms

$$\begin{aligned} \Lambda_i - \Lambda_{i-1} &= \mu_{i,1}\nu_{i,1} + \nu_{i,2} \\ \nu_{i,1} &= \mu_{i,2}\nu_{i,2} + \nu_{i,3} \\ &\dots \\ \nu_{i,\omega(i)-1} &= \mu_{i,\omega(1)}\nu_{i,\omega(i)} \end{aligned}$$

for $i = 1, \dots g$, and $\omega(i)$ is the number of steps before the Euclidean algorithm terminates and $\Lambda_0 = 0$. Then, the multiplicity $\nu_{i,j}$ appears $\mu_{i,j}$ times in the multiplicity sequence. Using this algorithm, we can reconstruct the exponents of the Puiseux expansion.

Using the same g Euclidean algorithms, we can construct the Resolution Graph as follows.

The graph consists of g chains P_i, called Puiseux chains, and each P_i is composed of elementary chains $\mathfrak{E}_{i,j}$.

These elementary chains $\mathfrak{E}_{i,j}$ represent blowing up the $\mu_{i,j}$ curves obtained from blowing up the $\mu_{i,j}$ curves with multiplicity $\nu_{i,j}$, where the weights of the graph are successive. Hence, we need only know the length and starting value and length of the chain.

The points of P_i are the points of the $\mathfrak{E}_{i,j}$, and we connect the points as follows: points within an $\mathfrak{E}_{i,j}$ are connected as in $\mathfrak{E}_{i,j}$, and the final point of $\mathfrak{E}_{i,j}$ is connected to the initial point of $\mathfrak{E}_{i,j+2}$, if it exists. The last point of $\mathfrak{E}j, \omega(i) - 1$ is connected to the final point of $\mathfrak{E}_{i,\omega(i)}$, called the contact point of P_i. The *initial point* of P_i is the initial point of $\mathfrak{E}_{i,1}$, or $\mathfrak{E}_{i,3}$ if $\nu_{i,1} = 0$ and $\omega(i) \geq 3$ and it is the final point of $\mathfrak{E}_{i,2}$. The end point of P_i is the initial point of $\mathfrak{E}_{i,2}$.

We connect the Puiseux chains as follows:

(a) if $i < g$, then the contact point of P_i is connected to the initial point of P_{i+1}

(b) the contact point of P_i is connected to the vertex representing the standard resolution.

Summarising we obtain:

Theorem 4.3.6. Consider a curve $C \subset \mathbb{CP}^2$, and a standard resolution, then

(a) Given the characteristic Puiseux exponents, we can construct the multiplicity sequence and the resolution graph.

(b) Given the multiplicity sequence, we can construct the characteristic exponents and the resolution graph.

(c) Given the resolution graph, we can construct the characteristic exponents and the multiplicity sequence.

Remark 4.3.7. The Puiseux characteristic exponents are an invariant of the singularity, and so any two standard resolutions of a singularity are equivalent.

In this chapter, we introduced two methods of describing a standard resolution of a singularity p of $C \subset \mathbb{CP}^2$: the multiplicity sequence and the resolution graph. We showed that we can use these to obtain the characteristic Puiseux exponents. The next steps in understanding the singularity p would be to examine the topological nature of p by looking at the intersection of C with an open ball centred at p. This would lead to examining the links and knots obtained, and allow for a classification of singularities.

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